Chapter 11
Output Analysis for a Single Model

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Discrete-Event System Simulation

Purpose

- Objective: Estimate system performance via simulation
- If θ is the system performance, the precision of the estimator $\hat{\theta}$ can be measured by:
  - The standard error of $\hat{\theta}$.
  - The width of a confidence interval (CI) for θ.
- Purpose of statistical analysis:
  - To estimate the standard error or CI .
  - To figure out the number of observations required to achieve desired error/CI.
- Potential issues to overcome:
  - Autocorrelation, e.g. inventory cost for subsequent weeks lack statistical independence.
  - Initial conditions, e.g. inventory on hand and # of backorders at time 0 would most likely influence the performance of week 1.

Outline

- Distinguish the two types of simulation: transient vs. steady state.
- Illustrate the inherent variability in a stochastic discrete-event simulation.
- Cover the statistical estimation of performance measures.
- Discusses the analysis of transient simulations.
- Discusses the analysis of steady-state simulations.

Type of Simulations

- Terminating verses non-terminating simulations
- Terminating simulation:
  - Runs for some duration of time T_E, where E is a specified event that stops the simulation.
  - Starts at time *0* under well-specified initial conditions.
  - Ends at the stopping time T_E.
  - Bank example: Opens at 8:30 am (time *0*) with no customers present and *8* of the *11* teller working (initial conditions), and closes at 4:30 pm (Time $T_E = 480$ minutes).
  - The simulation analyst chooses to consider it a terminating system because the object of interest is one day's operation.

Type of Simulations

- Non-terminating simulation:
  - Runs continuously, or at least over a very long period of time.
  - Examples: assembly lines that shut down infrequently, telephone systems, hospital emergency rooms.
  - Initial conditions defined by the analyst.
  - Runs for some analyst-specified period of time T_E.
  - Study the steady-state (long-run) properties of the system, properties that are not influenced by the initial conditions of the model.
- Whether a simulation is considered to be terminating or non-terminating depends on both
  - The objectives of the simulation study and
  - The nature of the system.

Stochastic Nature of Output Data

- Model output consist of one or more random variables (r. v.) because the model is an input-output transformation and the input variables are r.v.'s.
- M/G/1 queueing example:
  - Poisson arrival rate = *0.1* per minute; service time ~ $N(\mu = 9.5, \sigma = 1.75)$.
  - System performance: long-run mean queue length, $L_Q(t)$.
  - Suppose we run a single simulation for a total of 5,000 minutes
    - Divide the time interval [*0, 5000*) into *5* equal subintervals of *1000* minutes.
    - Average number of customers in queue from time *(j-1)1000* to *j(1000)* is Y_j.

Stochastic Nature of Output Data

- M/G/1 queueing example (cont.):
  - Batched average queue length for 3 independent replications:

Batching Interval (minutes)	Batch, j	Replication 1, Y_{1j}	Replication 2, Y_{2j}	Replication 3, Y_{3j}
[0, 1000)	1	3.61	2.91	7.67
[1000, 2000)	2	3.21	9.00	19.53
[2000, 3000)	3	2.18	16.15	20.36
[3000, 4000)	4	6.92	24.53	8.11
[4000, 5000)	5	2.82	25.19	12.62
[0, 5000)		3.75	15.56	13.66

  - Inherent variability in stochastic simulation both within a single replication and across different replications.
  - The average across 3 replications, $\bar{Y}_{1.}, \bar{Y}_{2.}, \bar{Y}_{3.}$ can be regarded as independent observations, but averages within a replication, Y_{11}, …, Y_{15}, are not.

Measures of performance

- Consider the estimation of a performance parameter, θ (or ϕ), of a simulated system.
  - Discrete time data: $[Y_1, Y_2, \ldots, Y_n]$, with ordinary mean: θ
  - Continuous-time data: $\{Y(t), 0 \leq t \leq T_E\}$ with time-weighted mean: ϕ
- Point estimation for discrete time data.
  - The point estimator:

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

    - Is unbiased if its expected value is θ, that is if: $E(\hat{\theta}) = \theta$ ← **_Desired_**
    - Is biased if: $E(\hat{\theta}) \neq \theta$

Point Estimator

- Point estimation for continuous-time data.
  - The point estimator:

$$\hat{\phi} = \frac{1}{T_E}\int_0^{T_E} Y(t)dt$$

    - Is biased in general where: $E(\hat{\phi}) \neq \phi$.
    - An unbiased or low-bias estimator is desired.
- Usually, system performance measures can be put into the common framework of θ or ϕ:
  - e.g., the proportion of days on which sales are lost through an out-of-stock situation, let:

$$Y(t) = \begin{cases} 1, & \text{if out of stock on day } i \\ 0, & \text{otherwise} \end{cases}$$

Point Estimator

[Performance Measures]

- Performance measure that does not fit: quantile or percentile: $\Pr\{Y \le \theta\} = p$
  - Estimating quantiles: the inverse of the problem of estimating a proportion or probability.
  - Consider a histogram of the observed values *Y*:
    - Find $\hat{\theta}$ such that *100p*% of the histogram is to the left of (smaller than) $\hat{\theta}$.

Confidence-Interval Estimation

[Performance Measures]

- To understand confidence intervals fully, it is important to distinguish between measures of error, and measures of risk, e.g., confidence interval versus prediction interval.
- Suppose the model is the normal distribution with mean θ, variance σ^2 (both unknown).
  - Let Y_i be the average cycle time for parts produced on the i^{th} replication of the simulation (its mathematical expectation is θ).
  - Average cycle time will vary from day to day, but over the long-run the average of the averages will be close to θ.
  - Sample variance across R replications: $S^2 = \frac{1}{R-1}\sum_{i=1}^{R}(Y_{i.} - \overline{Y}_{..})^2$

Confidence-Interval Estimation

[Performance Measures]

- Confidence Interval (CI):
  - A measure of error.
  - Where $Y_{i.}$ are normally distributed.

$$\bar{Y}_{..} \pm t_{\alpha/2,R-1} \frac{S}{\sqrt{R}}$$

  - We cannot know for certain how far $\bar{Y}_{..}$ is from θ but CI attempts to bound that error.
  - A CI, such as 95%, tells us how much we can trust the interval to actually bound the error between $\bar{Y}_{..}$ and θ.
  - The more replications we make, the less error there is in $\bar{Y}_{..}$ (converging to *0* as *R* goes to infinity).

Confidence-Interval Estimation

[Performance Measures]

- Prediction Interval (PI):
  - A measure of risk.
  - A good guess for the average cycle time on a particular day is our estimator but it is unlikely to be exactly right.
  - PI is designed to be wide enough to contain the *actual* average cycle time on any particular day with high probability.
  - Normal-theory prediction interval:

$$\bar{Y}_{..} \pm t_{\alpha/2, R-1} S \sqrt{1 + \frac{1}{R}}$$

  - The length of PI will not go to *0* as *R* increases because we can never simulate away risk.
  - PI's limit is: $\theta \pm z_{\alpha/2}\sigma$

Output Analysis for Terminating Simulations

- A terminating simulation: runs over a simulated time interval [$0, T_E$].
- A common goal is to estimate:

$$\theta = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right), \qquad \text{for discrete output}$$

$$\phi = E\left(\frac{1}{T_E}\int_0^{T_E} Y(t)\,dt\right), \quad \text{for continuous output } Y(t), 0 \le t \le T_E$$

- In general, independent replications are used, each run using a different random number stream and independently chosen initial conditions.

Statistical Background

[Terminating Simulations]

- Important to distinguish within-replication data from across-replication data.
- For example, simulation of a manufacturing system
  - Two performance measures of that system: cycle time for parts and work in process (WIP).
  - Let Y_{ij} be the cycle time for the j^{th} part produced in the i^{th} replication.
  - Across-replication data are formed by summarizing within-replication data $\bar{Y}_{i.}$.

Statistical Background

[Terminating Simulations]

- Across Replication:
  - For example: the daily cycle time averages (discrete time data)
    - The average: $\bar{Y}_{..} = \frac{1}{R}\sum_{i=1}^{R} Y_{i.}$
    - The sample variance: $S^2 = \frac{1}{R-1}\sum_{i=1}^{R}(Y_{i.} - \bar{Y}_{..})^2$
    - The confidence-interval half-width: $H = t_{\alpha/2,R-1}\frac{S}{\sqrt{R}}$
- Within replication:
  - For example: the WIP (a continuous time data)
    - The average: $\bar{Y}_{i.} = \frac{1}{T_{Ei}}\int_0^{T_{Ei}} Y_i(t)dt$
    - The sample variance: $S_i^2 = \frac{1}{T_{Ei}}\int_0^{T_{Ei}} \left(Y_i(t) - \bar{Y}_{i.}\right)^2 dt$

Statistical Background

[Terminating Simulations]

- Overall sample average, $\bar{Y}_{..}$, and the interval replication sample averages, $\bar{Y}_{i.}$, are always unbiased estimators of the expected daily average cycle time or daily average WIP.

- Across-replication data are independent (different random numbers) and identically distributed (same model), but within-replication data do not have these properties.

C.I. with Specified Precision

[Terminating Simulations]

- The half-length *H* of a *100(1 – α)%* confidence interval for a mean θ, based on the *t* distribution, is given by:

$$H = t_{\alpha/2,R-1}\frac{S}{\sqrt{R}}$$

R is the # of replications

S^2 is the sample variance

- Suppose that an error criterion ε is specified with probability *1 - α*, a sufficiently large sample size should satisfy:

$$P\left(\left|\bar{Y}_{..} - \theta\right| < \varepsilon\right) \geq 1 - \alpha$$

C.I. with Specified Precision

[Terminating Simulations]

- Assume that an initial sample of size R_0 (independent) replications has been observed.
- Obtain an initial estimate S_0^2 of the population variance σ^2.
- Then, choose sample size R such that $R \geq R_0$:
  - Since $t_{\alpha/2, R-1} \geq z_{\alpha/2}$, an initial estimate of R:

$$R \geq \left(\frac{z_{\alpha/2} S_0}{\varepsilon} \right)^2, \quad z_{\alpha/2} \text{ is the standard normal distribution.}$$

  - R is the smallest integer satisfying $R \geq R_0$ and $R \geq \left(\frac{t_{\alpha/2, R-1} S_0}{\varepsilon} \right)^2$
- Collect $R - R_0$ additional observations.
- The $100(1-\alpha)\%$ C.I. for θ:

$$\bar{Y}_{..} \pm t_{\alpha/2, R-1} \frac{S}{\sqrt{R}}$$

C.I. with Specified Precision

[Terminating Simulations]

- Call Center Example: estimate the agent's utilization ρ over the first 2 hours of the workday.
  - Initial sample of size $R_0 = 4$ is taken and an initial estimate of the population variance is $S_0^2 = (0.072)^2 = 0.00518$.
  - The error criterion is $\varepsilon = 0.04$ and confidence coefficient is $1-\alpha = 0.95$, hence, the final sample size must be at least:

$$\left(\frac{z_{0.025}S_0}{\varepsilon}\right)^2 = \frac{1.96^2 * 0.00518}{0.04^2} = 12.14$$

  - For the final sample size:

R	13	14	15
$t_{0.025,\ R-1}$	2.18	2.16	2.14
$\left(t_{\alpha/2,R-1}S_0 / \varepsilon\right)^2$	15.39	15.1	14.83

  - $R = 15$ is the smallest integer satisfying the error criterion, so $R - R_0 = 11$ additional replications are needed.
  - After obtaining additional outputs, half-width should be checked.

Quantiles

[Terminating Simulations]

- In this book, a proportion or probability is treated as a special case of a mean.
- When the number of independent replications $Y_1, \ldots, Y_R$ is large enough that $t_{\alpha/2,n-1} = z_{\alpha/2}$, the confidence interval for a probability p is often written as:

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{R-1}}$$

The sample proportion

- A quantile is the inverse of the probability to the probability estimation problem:

p is given

Find θ such that $Pr(Y \le \theta) = p$

Quantiles

- The best way is to sort the outputs and use the $(R*p)^{th}$ smallest value, i.e., find θ such that *100p%* of the data in a histogram of *Y* is to the left of θ.
  - Example: If we have *R=10* replications and we want the *p* = 0.8 quantile, first sort, then estimate θ by the *(10)(0.8)* = 8^{th} smallest value (round if necessary).

5.6 ←sorted data
7.1
8.8
8.9
9.5
9.7
10.1
12.2 ←this is our point estimate
12.5
12.9

Quantiles

- Confidence Interval of Quantiles: An approximate $(1-\alpha)100\%$ confidence interval for θ can be obtained by finding two values θ_l and θ_u.
  - θ_l cuts off $100p_l\%$ of the histogram (the Rp_l smallest value of the sorted data).
  - θ_u cuts off $100p_u\%$ of the histogram (the Rp_u smallest value of the sorted data).

$$\text{where } p_\ell = p - z_{\alpha/2}\sqrt{\frac{p(1-p)}{R-1}}$$

$$p_u = p + z_{\alpha/2}\sqrt{\frac{p(1-p)}{R-1}}$$

Quantiles

- Example: Suppose *R = 1000* reps, to estimate the *p = 0.8* quantile with a *95%* confidence interval.
  - First, sort the data from smallest to largest.
  - Then estimate of θ by the *(1000)(0.8) = 800th* smallest value, and the point estimate is *212.03.*
  - And find the confidence interval:

$$p_\ell = 0.8 - 1.96\sqrt{\frac{.8(1-.8)}{1000-1}} = 0.78$$

$$p_u = 0.8 + 1.96\sqrt{\frac{.8(1-.8)}{1000-1}} = 0.82$$

$$\text{The c.i. is the } 780^{\text{th}} \text{ and } 820^{\text{th}} \text{ smallest values}$$

A portion of the *1000* sorted values:

Output	Rank
180.92	779
188.96	**780**
190.55	781
208.58	799
212.03	**800**
216.99	801
250.32	819
256.79	**820**
256.99	821

  - The point estimate is The 95% c.i. is [188.96, 256.79]

Output Analysis for Steady-State Simulation

- Consider a single run of a simulation model to estimate a steady-state or long-run characteristics of the system.
  - The single run produces observations Y_1, Y_2, ... (generally the samples of an autocorrelated time series).
  - Performance measure:

$$\theta = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Y_i, \quad \text{for discrete measure} \quad \text{(with probability 1)}$$

$$\phi = \lim_{T_E \to \infty} \frac{1}{T_E} \int_0^{T_E} Y(t)dt, \quad \text{for continuous measure} \quad \text{(with probability 1)}$$

- Independent of the initial conditions.

Output Analysis for Steady-State Simulation

- The sample size is a design choice, with several considerations in mind:
  - Any bias in the point estimator that is due to artificial or arbitrary initial conditions (bias can be severe if run length is too short).
  - Desired precision of the point estimator.
  - Budget constraints on computer resources.
- Notation: the estimation of θ from a discrete-time output process.
  - One replication (or run), the output data: $Y_1, Y_2, Y_3, \ldots$
  - With several replications, the output data for replication r: $Y_{r1}, Y_{r2}, Y_{r3}, \ldots$

Initialization Bias [Steady-State Simulations]

- Methods to reduce the point-estimator bias caused by using artificial and unrealistic initial conditions:
  - Intelligent initialization.
  - Divide simulation into an initialization phase and data-collection phase.
- Intelligent initialization
  - Initialize the simulation in a state that is more representative of long-run conditions.
  - If the system exists, collect data on it and use these data to specify more nearly typical initial conditions.
  - If the system can be simplified enough to make it mathematically solvable, e.g. queueing models, solve the simplified model to find long-run expected or most likely conditions, use that to initialize the simulation.

Initialization Bias [Steady-State Simulations]

- Divide each simulation into two phases:
  - An initialization phase, from time *0* to time T_0.
  - A data-collection phase, from T_0 to the stopping time T_0+T_E.
  - The choice of T_0 is important:
    - After T_0, system should be more nearly representative of steady-state behavior.
  - System has reached steady state: the probability distribution of the system state is close to the steady-state probability distribution (bias of response variable is negligible).

Initialization Bias [Steady-State Simulations]

- M/G/1 queueing example: A total of 10 independent replications were made.
  - Each replication beginning in the empty and idle state.
  - Simulation run length on each replication was T_0+T_E = 15,000 minutes.
  - Response variable: queue length, $L_Q(t,r)$ (at time t of the rth replication).
  - Batching intervals of 1,000 minutes, batch means
- Ensemble averages:
  - To identify trend in the data due to initialization bias
  - The average corresponding batch means *across* replications:

$$\bar{Y}_{.j} = \frac{1}{R}\sum_{r=1}^{R} Y_{rj}$$

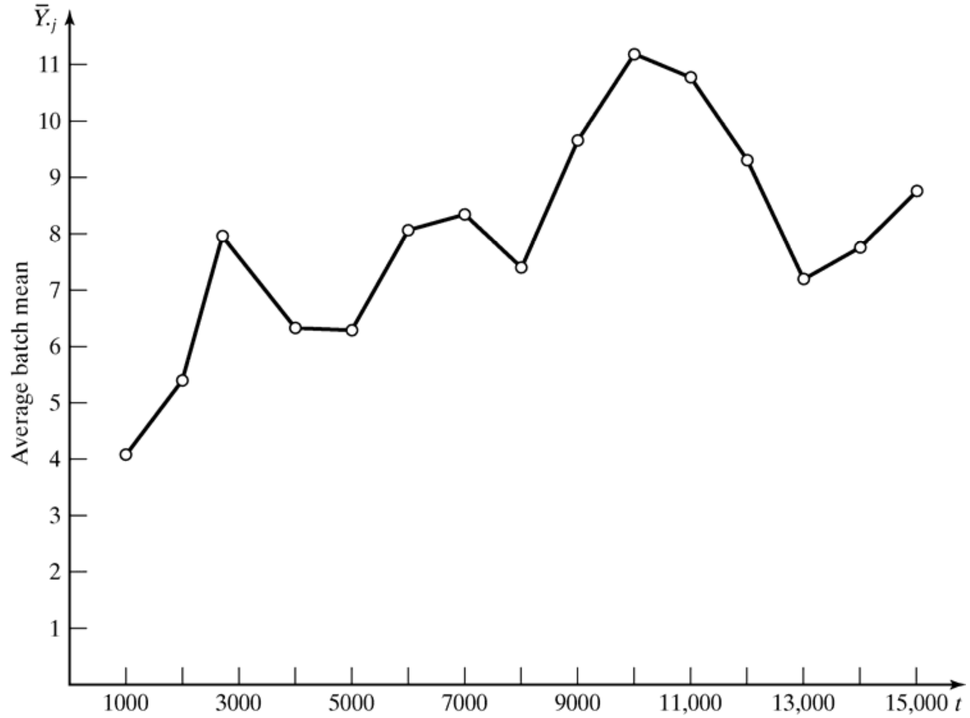
R replications

  - The preferred method to determine deletion point.

Initialization Bias

[Steady-State Simulations]

- A plot of the ensemble averages, $\bar{Y}_{\cdot\cdot}(n,d)$, versus *1000j*, for *j = 1,2, …,15*.



  - Illustrates the downward bias of the initial observations.

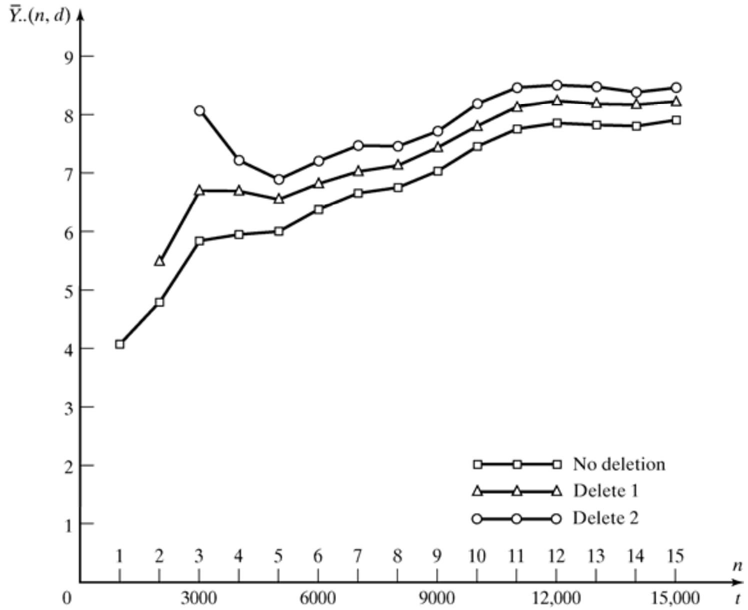
Initialization Bias

[Steady-State Simulations]

- Cumulative average sample mean (after deleting *d* observations):

$$\bar{Y}_{..}(n,d) = \frac{1}{n-d}\sum_{j=d+1}^{n}\bar{Y}_{.j}$$

  - Not recommended to determine the initialization phase.



  - It is apparent that downward bias is present and this bias can be reduced by deletion of one or more observations.

Initialization Bias

[Steady-State Simulations]

- No widely accepted, objective and proven technique to guide how much data to delete to reduce initialization bias to a negligible level.
- Plots can, at times, be misleading but they are still recommended.
  - Ensemble averages reveal a smoother and more precise trend as the # of replications, R, increases.
  - Ensemble averages can be smoothed further by plotting a moving average.
  - Cumulative average becomes less variable as more data are averaged.
  - The more correlation present, the longer it takes for $\bar{Y}_{.j}$ to approach steady state.
  - Different performance measures could approach steady state at different rates.

Error Estimation

[Steady-State Simulations]

- If $\{Y_1, \ldots, Y_n\}$ are not statistically independent, then S^2/n is a biased estimator of the true variance.
  - Almost always the case when $\{Y_1, \ldots, Y_n\}$ is a sequence of output observations from within a single replication (autocorrelated sequence, time-series).
- Suppose the point estimator θ is the sample mean

$$\bar{Y} = \sum_{i=1}^{n} Y_i / n$$

  - Variance of $\bar{Y}$ is almost impossible to estimate.
  - For system with steady state, produce an output process that is approximately covariance stationary (after passing the transient phase).
    - The covariance between two random variables in the time series depends only on the lag (the # of observations between them).

Error Estimation [Steady-State Simulations]

- For a covariance stationary time series, $\{Y_1, \ldots, Y_n\}$:
  - Lag-k autocovariance is: $\delta_k = \text{cov}(Y_1, Y_{1+k}) = \text{cov}(Y_i, Y_{i+k})$
  - Lag-k autocorrelation is: $\rho_k = \frac{\gamma_k}{\sigma^2}$
- If a time series is covariance stationary, then the variance of $\bar{Y}$ is:

$$V(\bar{Y}) = \frac{\sigma^2}{n}\underbrace{\left[1 + 2\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\rho_k\right]}_{c}$$

- The expected value of the variance estimator is:

$$E\left(\frac{S^2}{n}\right) = BV(\bar{Y}), \qquad \text{where } B = \frac{n/c - 1}{n - 1}$$

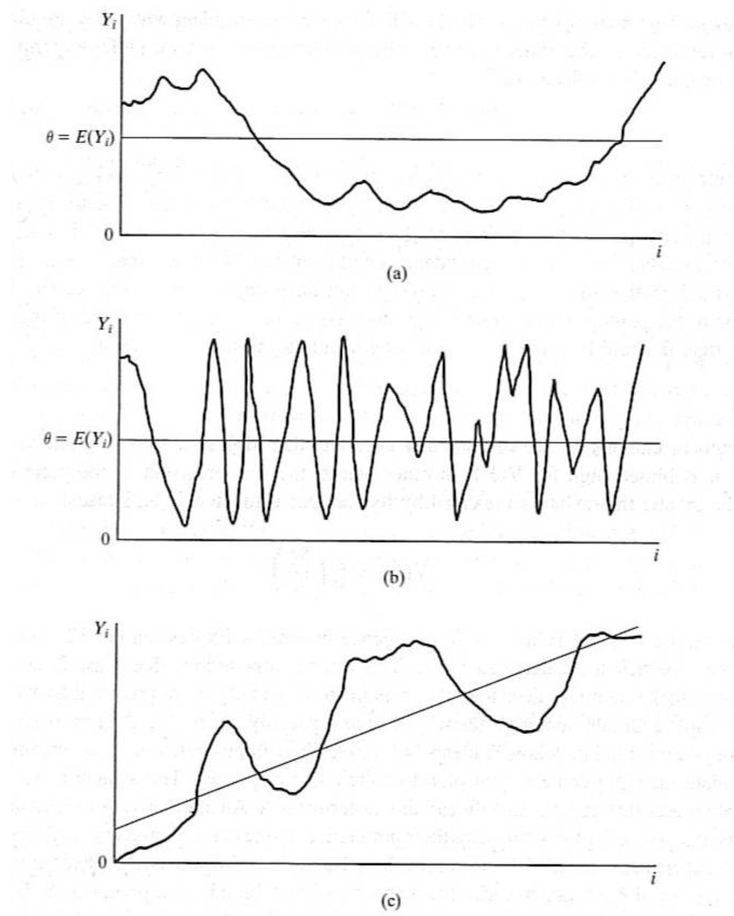
Error Estimation

[Steady-State Simulations]

a) Stationary time series Y_i exhibiting positive autocorrelation.

b) Stationary time series Y_i exhibiting negative autocorrelation.

c) Nonstationary time series with an upward trend



Error Estimation

[Steady-State Simulations]

- The expected value of the variance estimator is:

$$E\left(\frac{S^2}{n}\right) = BV(\bar{Y}), \qquad \text{where } B = \frac{n/c-1}{n-1} \text{ and } V(\bar{Y}) \text{ is the variance of } \bar{Y}$$

  - If Y_i are independent, then S^2/n is an unbiased estimator of $V(\bar{Y})$
  - If the autocorrelation ρ_k are primarily positive, then S^2/n is biased low as an estimator of $V(\bar{Y})$.
  - If the autocorrelation ρ_k are primarily negative, then S^2/n is biased high as an estimator of $V(\bar{Y})$.

Replication Method [Steady-State Simulations]

- Use to estimate point-estimator variability and to construct a confidence interval.
- Approach: make R replications, initializing and deleting from each one the same way.
- Important to do a thorough job of investigating the initial-condition bias:
  - Bias is not affected by the number of replications, instead, it is affected only by deleting more data (i.e., increasing T_0) or extending the length of each run (i.e. increasing T_E).
- Basic raw output data $\{Y_{rj}, r = 1, ..., R; j = 1, \ldots, n\}$ is derived by:
  - Individual observation from within replication *r*.
  - Batch mean from within replication *r* of some number of discrete-time observations.
  - Batch mean of a continuous-time process over time interval *j*.

Replication Method [Steady-State Simulations]

- Each replication is regarded as a single sample for estimating θ. For replication *r:* $\bar{Y}_{r.}(n,d) = \frac{1}{n-d}\sum_{j=d+1}^{n} Y_{rj}$
- The overall point estimator:

$$\bar{Y}_{..}(n,d) = \frac{1}{R}\sum_{r=1}^{R}\bar{Y}_{r.}(n,d) \quad \text{and} \quad \mathrm{E}[\bar{Y}_{..}(n,d)] = \theta_{n,d}$$

- If *d* and *n* are chosen sufficiently large:
  - $\theta_{n,d} \sim \theta$.
  - $\bar{Y}_{..}(n,d)$ is an approximately unbiased estimator of θ.
- To estimate standard error of $\bar{Y}_{..}$, the sample variance and standard error:

$$S^2 = \frac{1}{R-1}\sum_{r=1}^{R}(\bar{Y}_{r.} - \bar{Y}_{..})^2 = \frac{1}{R-1}\left(\sum_{r=1}^{R}\bar{Y}_{r.}^{\,2} - R\bar{Y}_{..}^{\,2}\right) \quad \text{and} \quad s.e.(\bar{Y}_{..}) = \frac{S}{\sqrt{R}}$$

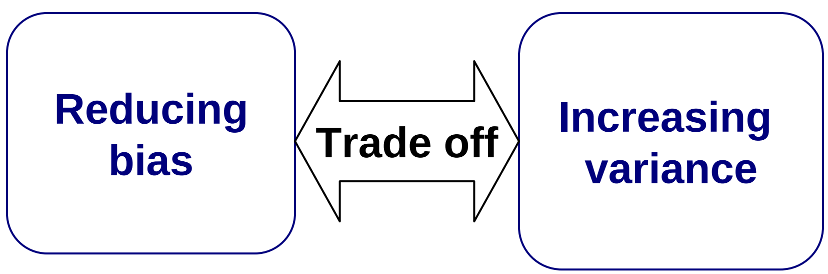
Replication Method

[Steady-State Simulations]

- Length of each replication (*n*) beyond deletion point (*d*):

  $$(n - d) > 10d$$

- Number of replications (*R*) should be as many as time permits, up to about *25* replications.
- For a fixed total sample size (*n*), as fewer data are deleted (*d*): ↓
  - C.I. shifts: greater bias.
  - Standard error of $\bar{Y}_{..}(n,d)$ decreases: decrease variance.



Replication Method [Steady-State Simulations]

- M/G/1 queueing example:
  - Suppose *R = 10*, each of length T_E = *15,000* minutes, starting at time *0* in the empty and idle state, initialized for T_0 = *2,000* minutes before data collection begins.
  - Each batch means is the average number of customers in queue for a *1,000*-minute interval.
  - The 1^{st} two batch means are deleted (*d = 2*).
  - The point estimator and standard error are:

$$\bar{Y}_{..}(15,2) = 8.43 \qquad \text{and} \qquad s.e.\left(\bar{Y}_{..}(15,2)\right) = 1.59$$

  - The *95%* C.I. for long-run mean queue length is:

$$\bar{Y}_{..} - t_{\alpha/2,R-1} S/\sqrt{R} \le \theta \le \bar{Y}_{..} + t_{\alpha/2,R-1} S/\sqrt{R}$$

$$8.43 - 2.26(1.59) \le L_Q \le 8.42 + 2.26(1.59)$$

  - A high degree of confidence that the long-run mean queue length is between *4.84* and *12.02* (if d and n are "large" enough).

Sample Size [Steady-State Simulations]

- To estimate a long-run performance measure, θ, within $\pm\varepsilon$ with confidence *100(1-α)%.*
- M/G/1 queueing example (cont.):
  - We know: $R_0 = 10$, $d = 2$ and $S_0^2 = 25.30$.
  - To estimate the long-run mean queue length, L_Q, within $\varepsilon = 2$ customers with *90%* confidence ($\alpha = 10\%$).
  - Initial estimate:

$$R \geq \left(\frac{z_{0.05} S_0}{\varepsilon}\right)^2 = \frac{1.645^2 (25.30)}{2^2} = 17.1$$

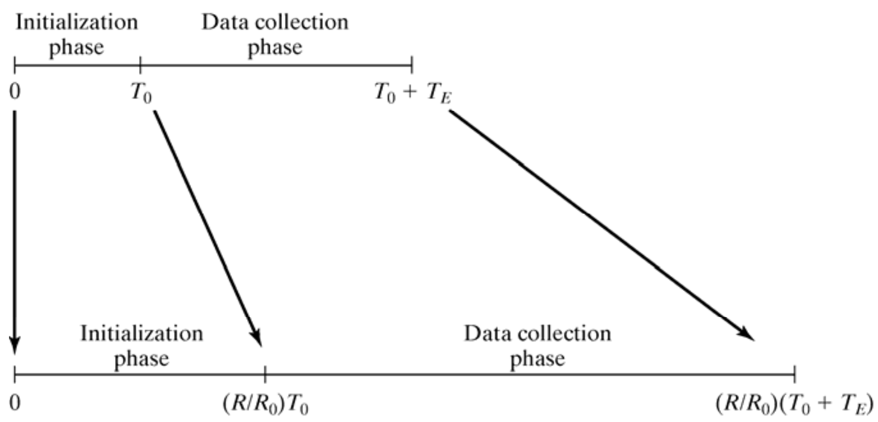
  - Hence, at least 18 replications are needed, next try $R = 18, 19, \ldots$ using $R \geq \left(t_{0.05,R-1} S_0 / \varepsilon\right)^2$. We found that:

$$R = 19 \geq \left(t_{0.05,19-1} S_0 / \varepsilon\right)^2 = (1.74 * 25.3 / 2)^2 = 18.93$$

  - Additional replications needed is $R - R_0 = 19\text{-}10 = 9$.

Sample Size [Steady-State Simulations]

- An alternative to increasing R is to increase total run length T_0+T_E within each replication.
  - Approach:
    - Increase run length from (T_0+T_E) to $(R/R_0)(T_0+T_E)$, and
    - Delete additional amount of data, from time 0 to time $(R/R_0)T_0$.
  - Advantage: any residual bias in the point estimator should be further reduced.
  - However, it is necessary to have saved the state of the model at time T_0+T_E and to be able to restart the model.



Batch Means for Interval Estimation

[Steady-State Simulations]

- Using a single, long replication:
  - Problem: data are dependent so the usual estimator is biased.
  - Solution: batch means.
- Batch means: divide the output data from *1* replication (after appropriate deletion) into a few large batches and then treat the means of these batches as if they were independent.
- A continuous-time process, $\{Y(t),\ T_0 \le t \le T_0+T_E\}$:
  - k batches of size $m = T_E/k$, batch means:

$$\bar{Y}_j = \frac{1}{m}\int_{(j-1)m}^{jm} Y(t+T_0)\,dt$$

- A discrete-time process, $\{Y_i,\ i = d+1, d+2, \ldots, n\}$:
  - k batches of size $m = (n-d)/k$, batch means: $\bar{Y}_j = \dfrac{1}{m}\displaystyle\sum_{i=(j-1)m+1}^{jm} Y_{i+d}$

Batch Means for Interval Estimation

[Steady-State Simulations]

$$\underbrace{Y_1, \ldots, Y_d}_{\text{deleted}}, \underbrace{Y_{d+1}, \ldots, Y_{d+m}}_{\bar{Y}_1}, \underbrace{Y_{d+m+1}, \ldots, Y_{d+2m}}_{\bar{Y}_2}, \ \cdots \ , \underbrace{Y_{d+(k-1)m+1}, \ldots, Y_{d+km}}_{\bar{Y}_k}$$

- Starting either with continuous-time or discrete-time data, the variance of the sample mean is estimated by:

$$\frac{S^2}{k} = \frac{1}{k}\sum_{j=1}^{k}\frac{\left(\bar{Y}_j - \bar{Y}\right)^2}{k-1} = \sum_{j=1}^{k}\frac{\bar{Y}_j^{\,2} - k\bar{Y}^2}{k(k-1)}$$

- If the batch size is sufficiently large, successive batch means will be approximately independent, and the variance estimator will be approximately unbiased.
- No widely accepted and relatively simple method for choosing an acceptable batch size *m* (see text for a suggested approach). Some simulation software does it automatically.

Summary

- Stochastic discrete-event simulation is a statistical experiment.
  - Purpose of statistical experiment: obtain estimates of the performance measures of the system.
  - Purpose of statistical analysis: acquire some assurance that these estimates are sufficiently precise.
- Distinguish: terminating simulations and steady-state simulations.
- Steady-state output data are more difficult to analyze
  - Decisions: initial conditions and run length
  - Possible solutions to bias: deletion of data and increasing run length
- Statistical precision of point estimators are estimated by standard-error or confidence interval
- Method of independent replications was emphasized.