

6

- Introduction
- Basic Proportionality Theorem
- Angle Bisector Theorem
- Similar Triangles
- Tangent chord theorem
- Pythagoras theorem



EUCLID
(300 BC)
Greece

Euclid's 'Elements' is one of the most influential works in the history of mathematics, serving as the main text book for teaching mathematics especially geometry.

Euclid's algorithm is an efficient method for computing the greatest common divisor.

GEOMETRY

There is geometry in the humming of the strings, there is music in the spacing of spheres - Pythagoras

6.1 Introduction

Geometry is a branch of mathematics that deals with the properties of various geometrical figures. The geometry which treats the properties and characteristics of various geometrical shapes with axioms or theorems, without the help of accurate measurements is known as theoretical geometry. The study of geometry improves one's power to think logically.

Euclid, who lived around 300 BC is considered to be the father of geometry. Euclid initiated a new way of thinking in the study of geometrical results by deductive reasoning based on previously proved results and some self evident specific assumptions called axioms or postulates.

Geometry holds a great deal of importance in fields such as engineering and architecture. For example, many bridges that play an important role in our lives make use of congruent and similar triangles. These triangles help to construct the bridge more stable and enables the bridge to withstand great amounts of stress and strain. In the construction of buildings, geometry can play two roles; one in making the structure more stable and the other in enhancing the beauty. Elegant use of geometric shapes can turn buildings and other structures such as the Taj Mahal into great landmarks admired by all. Geometric proofs play a vital role in the expansion and understanding of many branches of mathematics.

The basic proportionality theorem is attributed to the famous Greek mathematician **Thales**. This theorem is also called Thales theorem.

To understand the basic proportionality theorem, let us perform the following activity.

Activity

Draw any angle XAY and mark points (say five points) P_1, P_2, D, P_3 and B on arm AX such that $AP_1 = P_1P_2 = P_2D = DP_3 = P_3B = 1$ unit (say).

Through B draw any line intersecting arm AY at C . Again through D draw a line parallel to BC to intersect AC at E .

Now $AD = AP_1 + P_1P_2 + P_2D = 3$ units

and $DB = DP_3 + P_3B = 2$ units

$\therefore \frac{AD}{DB} = \frac{3}{2}$

Measure AE and EC .

We observe that $\frac{AE}{EC} = \frac{3}{2}$

Thus, in $\triangle ABC$ if $DE \parallel BC$, then $\frac{AD}{DB} = \frac{AE}{EC}$

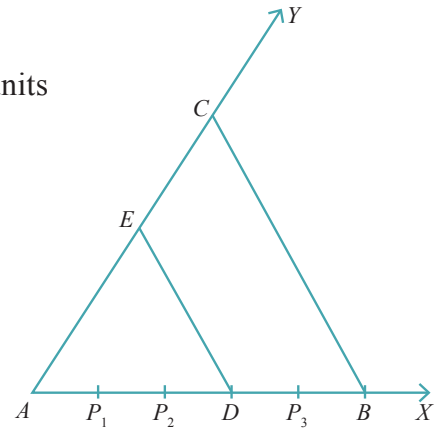


Fig. 6.1

We prove this result as a theorem known as Basic Proportionality Theorem or Thales Theorem as follows:

6.2 Basic proportionality and Angle Bisector theorems

Theorem 6.1 Basic Proportionality theorem or Thales Theorem

If a straight line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio.

Given: In a triangle ABC , a straight line l parallel to BC , intersects AB at D and AC at E .

To prove: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction: Join BE, CD .

Draw $EF \perp AB$ and $DG \perp CA$.

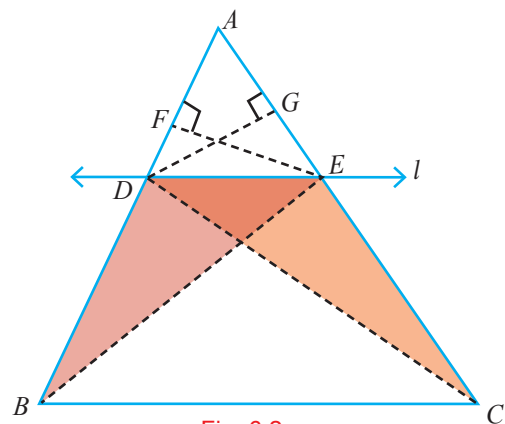


Fig. 6.2

Proof

Since, $EF \perp AB$, EF is the height of triangles ADE and DBE .

Area ($\triangle ADE$) = $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} AD \times EF$ and

Area ($\triangle DBE$) = $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} DB \times EF$

$$\therefore \frac{\text{area}(\triangle ADE)}{\text{area}(\triangle DBE)} = \frac{\frac{1}{2}AD \times EF}{\frac{1}{2}DB \times EF} = \frac{AD}{DB} \quad (1)$$

Similarly, we get

$$\frac{\text{area}(\triangle ADE)}{\text{area}(\triangle DCE)} = \frac{\frac{1}{2} \times AE \times DG}{\frac{1}{2} \times EC \times DG} = \frac{AE}{EC} \quad (2)$$

But, $\triangle DBE$ and $\triangle DCE$ are on the same base DE and between the same parallel straight lines BC and DE .

$$\therefore \text{area}(\triangle DBE) = \text{area}(\triangle DCE) \quad (3)$$

From (1), (2) and (3), we obtain $\frac{AD}{DB} = \frac{AE}{EC}$. Hence the theorem.

Corollary

If in a $\triangle ABC$, a straight line DE parallel to BC , intersects AB at D and AC at E , then

$$(i) \frac{AB}{AD} = \frac{AC}{AE} \quad (ii) \frac{AB}{DB} = \frac{AC}{EC}$$

Proof

(i) From Thales theorem, we have

$$\begin{aligned} \frac{AD}{DB} &= \frac{AE}{EC} \\ \Rightarrow \frac{DB}{AD} &= \frac{EC}{AE} \\ \Rightarrow 1 + \frac{DB}{AD} &= 1 + \frac{EC}{AE} \\ \Rightarrow \frac{AD + DB}{AD} &= \frac{AE + EC}{AE} \end{aligned}$$

$$\text{Thus, } \frac{AB}{AD} = \frac{AC}{AE}$$

(ii) Similarly, we can prove

$$\frac{AB}{DB} = \frac{AC}{EC}$$

Do you know?

$$\text{If } \frac{a}{b} = \frac{c}{d} \text{ then } \frac{a+b}{b} = \frac{c+d}{d}.$$

This is called componendo rule.

$$\text{Here, } \frac{DB}{AD} = \frac{EC}{AE}$$

$$\Rightarrow \frac{AD + DB}{AD} = \frac{AE + EC}{AE}$$

by componendo rule.

Is the converse of this theorem also true? To examine this let us perform the following activity.

Activity

Draw an angle $\angle XAY$ and on the ray AX , mark points P_1, P_2, P_3, P_4 and B such that $AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4B = 1$ unit (say).

Similarly, on ray AY , mark points Q_1, Q_2, Q_3, Q_4 and C such that

$AQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4 = Q_4C = 2$ units (say).

Now join P_1Q_1 and BC .

Then
$$\frac{AP_1}{P_1B} = \frac{1}{4} \quad \text{and} \quad \frac{AQ_1}{Q_1C} = \frac{2}{8} = \frac{1}{4}$$

Thus,
$$\frac{AP_1}{P_1B} = \frac{AQ_1}{Q_1C}$$

We observe that the lines P_1Q_1 and BC are parallel to each other. i.e., $P_1Q_1 \parallel BC$

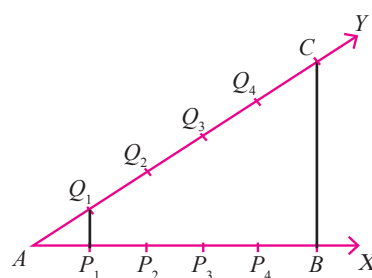


Fig. 6.3 (1)

Similarly, by joining P_2Q_2 , P_3Q_3 and P_4Q_4 we see that

$$\frac{AP_2}{P_2B} = \frac{AQ_2}{Q_2C} = \frac{2}{3} \quad \text{and} \quad P_2Q_2 \parallel BC \quad (2)$$

$$\frac{AP_3}{P_3B} = \frac{AQ_3}{Q_3C} = \frac{3}{2} \quad \text{and} \quad P_3Q_3 \parallel BC \quad (3)$$

$$\frac{AP_4}{P_4B} = \frac{AQ_4}{Q_4C} = \frac{4}{1} \quad \text{and} \quad P_4Q_4 \parallel BC \quad (4)$$

From (1), (2), (3) and (4) we observe that if a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.

In this direction, let us state and prove a theorem which is the converse of Thales theorem.

Theorem 6.2

Converse of Basic Proportionality Theorem (Converse of Thales Theorem)

If a straight line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.

Given: A line l intersects the sides AB and AC of $\triangle ABC$ respectively at D and E

such that
$$\frac{AD}{DB} = \frac{AE}{EC} \quad (1)$$

To prove : $DE \parallel BC$

Construction : If DE is not parallel to BC , then draw a line $DF \parallel BC$.

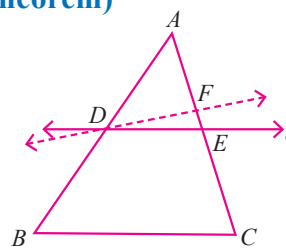


Fig. 6.4

Proof Since $DF \parallel BC$, by Thales theorem we get,

$$\frac{AD}{DB} = \frac{AF}{FC} \quad (2)$$

From (1) and (2), we get
$$\frac{AF}{FC} = \frac{AE}{EC} \implies \frac{AF + FC}{FC} = \frac{AE + EC}{EC}$$

$$\frac{AC}{FC} = \frac{AC}{EC} \quad \therefore FC = EC$$

This is possible only when F and E coincide. Thus, $DE \parallel BC$.

Theorem 6.3**Angle Bisector Theorem**

The internal (external) bisector of an angle of a triangle divides the opposite side internally (externally) in the ratio of the corresponding sides containing the angle.

Case (i) (Internally)

Given : In $\triangle ABC$, AD is the internal bisector of $\angle BAC$ which meets BC at D .

To prove : $\frac{BD}{DC} = \frac{AB}{AC}$

Construction : Draw $CE \parallel DA$ to meet BA produced at E .

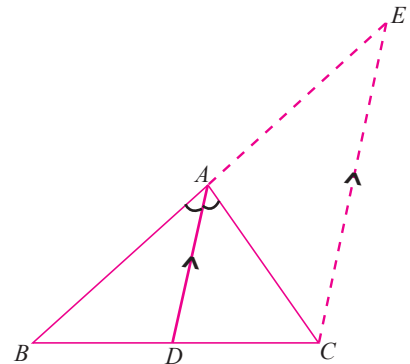


Fig. 6.5

Proof

Since $CE \parallel DA$ and AC is the transversal, we have

$$\angle DAC = \angle ACE \text{ (alternate angles)} \quad (1)$$

$$\text{and } \angle BAD = \angle AEC \text{ (corresponding angles)} \quad (2)$$

$$\text{Since } AD \text{ is the angle bisector of } \angle A, \angle BAD = \angle DAC \quad (3)$$

From (1), (2) and (3), we have $\angle ACE = \angle AEC$

Thus in $\triangle ACE$, we have $AE = AC$ (sides opposite to equal angles are equal)

Now in $\triangle BCE$ we have, $CE \parallel DA$

$$\frac{BD}{DC} = \frac{BA}{AE} \quad (\text{Thales theorem})$$

$$\Rightarrow \frac{BD}{DC} = \frac{AB}{AC} \quad (AE = AC)$$

Hence the theorem.

Case (ii) Externally (this part is not for examination)

Given : In $\triangle ABC$,
 AD is the external bisector of $\angle BAC$
and intersects BC produced at D .

To prove : $\frac{BD}{DC} = \frac{AB}{AC}$

Construction : Draw $CE \parallel DA$ meeting AB at E .

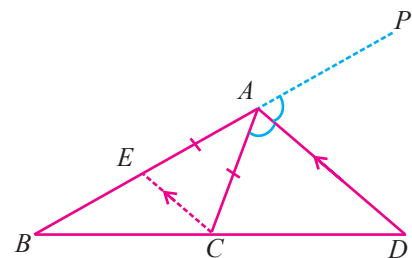


Fig. 6.6

Proof

$CE \parallel DA$ and AC is a transversal,

$$\angle ECA = \angle CAD \text{ (alternate angles)} \quad (1)$$

Also $CE \parallel DA$ and BP is a transversal

$$\angle CEA = \angle DAP \quad (\text{corresponding angles}) \quad (2)$$

But AD is the bisector of $\angle CAP$

$$\angle CAD = \angle DAP \quad (3)$$

From (1), (2) and (3), we have

$$\angle CEA = \angle ECA$$

Thus, in $\triangle ECA$, we have $AC = AE$ (sides opposite to equal angles are equal)

In $\triangle BDA$, we have $EC \parallel AD$

$$\therefore \frac{BD}{DC} = \frac{BA}{AE} \quad (\text{Thales theorem})$$

$$\Rightarrow \frac{BD}{DC} = \frac{BA}{AC} \quad (AE = AC)$$

Hence the theorem.

Theorem 6.4 Converse of Angle Bisector Theorem

If a straight line through one vertex of a triangle divides the opposite side internally (externally) in the ratio of the other two sides, then the line bisects the angle internally (externally) at the vertex.

Case (i) : (Internally)

Given : In $\triangle ABC$, the line AD divides the opposite side BC internally such that

$$\frac{BD}{DC} = \frac{AB}{AC}$$

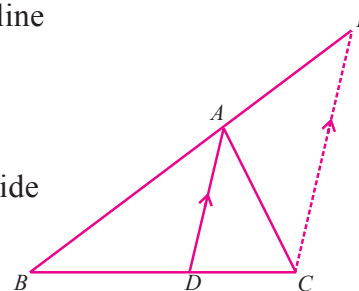


Fig. 6.7 (1)

To prove : AD is the internal bisector of $\angle BAC$.

i.e., to prove $\angle BAD = \angle DAC$.

Construction :

Through C draw $CE \parallel AD$ meeting BA produced at E .

Proof Since $CE \parallel AD$, by Thales theorem, we have $\frac{BD}{DC} = \frac{BA}{AE}$ (2)

Thus, from (1) and (2) we have, $\frac{AB}{AE} = \frac{AB}{AC}$

$$\therefore AE = AC$$

Now, in $\triangle ACE$, we have $\angle ACE = \angle AEC$ ($AE = AC$) (3)

Since AC is a transversal of the parallel lines AD and CE ,

we get, $\angle DAC = \angle ACE$ (alternate interior angles are equal) (4)

Also BE is a transversal of the parallel lines AD and CE .

we get $\angle BAD = \angle AEC$ (corresponding angles are equal) (5)

From (3), (4) and (5), we get

$$\angle BAD = \angle DAC$$

$\therefore AD$ is the angle bisector of $\angle BAC$.

Hence the theorem.

Case (ii) Externally (this part is not for examination)

Given : In $\triangle ABC$, the line AD divides externally the opposite side BC produced at D .

$$\text{such that } \frac{BD}{DC} = \frac{AB}{AC}$$

To prove : AD is the bisector of $\angle PAC$,

i.e., to prove $\angle PAD = \angle DAC$

Construction : Through C draw $CE \parallel DA$ meeting BA at E .

Proof Since $CE \parallel DA$, by Thales theorem $\frac{BD}{DC} = \frac{BA}{EA}$ (2)

From (1) and (2), we have

$$\frac{AB}{AE} = \frac{AB}{AC} \quad \therefore AE = AC$$

In $\triangle ACE$, we have $\angle ACE = \angle AEC$ ($AE = AC$) (3)

Since AC is a transversal of the parallel lines AD and CE , we have

$$\angle ACE = \angle DAC \quad (\text{alternate interior angles}) \quad (4)$$

Also, BA is a transversal of the parallel lines AD and CE ,

$$\angle PAD = \angle AEC \quad (\text{corresponding angles}) \quad (5)$$

From (3), (4) and (5), we get

$$\angle PAD = \angle DAC$$

$\therefore AD$ is the bisector of $\angle PAC$. Thus AD is the external bisector of $\angle BAC$

Hence the theorem.

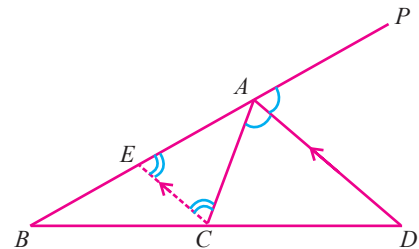


Fig. 6.8

(1)

Example 6.1

In $\triangle ABC$, $DE \parallel BC$ and $\frac{AD}{DB} = \frac{2}{3}$. If $AE = 3.7\text{cm}$, find EC .

Solution In $\triangle ABC$, $DE \parallel BC$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC} \quad (\text{Thales theorem})$$

$$\implies EC = \frac{AE \times DB}{AD}$$

$$\text{Thus, } EC = \frac{3.7 \times 3}{2} = 5.55\text{ cm}$$

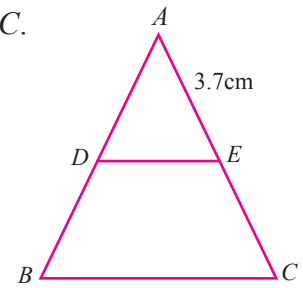


Fig. 6.9

Example 6.2

In $\triangle PQR$, given that S is a point on PQ such that

$ST \parallel QR$ and $\frac{PS}{SQ} = \frac{3}{5}$. If $PR = 5.6\text{ cm}$, then find PT .

Solution In $\triangle PQR$, we have $ST \parallel QR$ and by Thales theorem,

$$\frac{PS}{SQ} = \frac{PT}{TR}$$

(1)

Let $PT = x$. Thus, $TR = PR - PT = 5.6 - x$.

From (1), we get $PT = TR \left(\frac{PS}{SQ} \right)$

$$x = (5.6 - x) \left(\frac{3}{5} \right)$$

$$5x = 16.8 - 3x$$

Thus, $x = \frac{16.8}{8} = 2.1$ That is, $PT = 2.1\text{ cm}$.

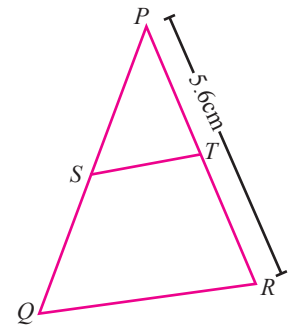


Fig. 6.10

Example 6.3

In a $\triangle ABC$, D and E are points on AB and AC respectively such that $\frac{AD}{DB} = \frac{AE}{EC}$ and $\angle ADE = \angle DEA$. Prove that $\triangle ABC$ is isosceles.

Solution Since $\frac{AD}{DB} = \frac{AE}{EC}$, by converse of Thales theorem, $DE \parallel BC$

$$\therefore \angle ADE = \angle ABC \quad \text{and} \quad (1)$$

$$\angle DEA = \angle BCA \quad (2)$$

But, given that $\angle ADE = \angle DEA$ (3)

From (1), (2) and (3), we get $\angle ABC = \angle BCA$

$\therefore AC = AB$ (If opposite angles are equal, then opposite sides are equal).

Thus, $\triangle ABC$ is isosceles.

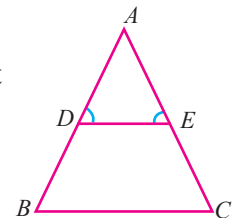


Fig. 6.11

Example 6.4

The points D , E and F are taken on the sides AB , BC and CA of a $\triangle ABC$ respectively, such that $DE \parallel AC$ and $FE \parallel AB$.

Prove that $\frac{AB}{AD} = \frac{AC}{FC}$

Solution Given that in $\triangle ABC$, $DE \parallel AC$.

$$\therefore \frac{BD}{DA} = \frac{BE}{EC} \quad (\text{Thales theorem}) \quad (1)$$

Also, given that $FE \parallel AB$.

$$\therefore \frac{BE}{EC} = \frac{AF}{FC} \quad (\text{Thales theorem}) \quad (2)$$

From (1) and (2), we get

$$\frac{BD}{AD} = \frac{AF}{FC}$$

$$\Rightarrow \frac{BD + AD}{AD} = \frac{AF + FC}{FC} \quad (\text{componendo rule})$$

$$\text{Thus, } \frac{AB}{AD} = \frac{AC}{FC}.$$

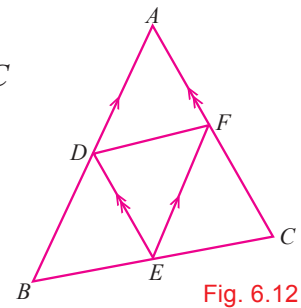


Fig. 6.12

Example 6.5

In $\triangle ABC$, the internal bisector AD of $\angle A$ meets the side BC at D . If $BD = 2.5$ cm, $AB = 5$ cm and $AC = 4.2$ cm, then find DC .

Solution In $\triangle ABC$, AD is the internal bisector of $\angle A$.

$$\therefore \frac{BD}{DC} = \frac{AB}{AC} \quad (\text{angle bisector theorem})$$

$$\Rightarrow DC = \frac{BD \times AC}{AB}$$

$$\text{Thus, } DC = \frac{2.5 \times 4.2}{5} = 2.1 \text{ cm.}$$

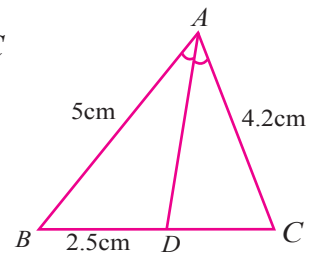


Fig. 6.13

Example 6.6

In $\triangle ABC$, AE is the external bisector of $\angle A$, meeting BC produced at E . If $AB = 10$ cm, $AC = 6$ cm and $BC = 12$ cm, then find CE .

Solution In $\triangle ABC$, AE is the external bisector of $\angle A$ meeting BC produced at E .

Let $CE = x$ cm. Now, by the angle bisector theorem, we have

$$\frac{BE}{CE} = \frac{AB}{AC} \Rightarrow \frac{12 + x}{x} = \frac{10}{6}$$

$$\Rightarrow 3(12 + x) = 5x.$$

$$\Rightarrow x = 18.$$

Hence, $CE = 18$ cm.

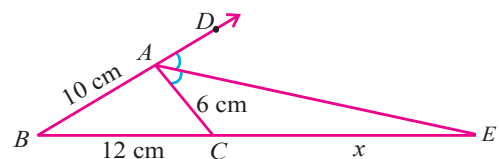


Fig. 6.14

Example 6.7

D is the midpoint of the side BC of $\triangle ABC$. If P and Q are points on AB and on AC such that DP bisects $\angle BDA$ and DQ bisects $\angle ADC$, then prove that $PQ \parallel BC$.

Solution In $\triangle ABD$, DP is the angle bisector of $\angle BDA$.

$$\therefore \frac{AP}{PB} = \frac{AD}{BD} \quad (\text{angle bisector theorem}) \quad (1)$$

In $\triangle ADC$, DQ is the bisector of $\angle ADC$

$$\therefore \frac{AQ}{QC} = \frac{AD}{DC} \quad (\text{angle bisector theorem}) \quad (2)$$

But, $BD = DC$ (D is the midpoint of BC)

$$\text{Now (2)} \implies \frac{AQ}{QC} = \frac{AD}{BD} \quad (3)$$

From (1) and (3) we get,

$$\frac{AP}{PB} = \frac{AQ}{QC}$$

Thus, $PQ \parallel BC$. (converse of Thales theorem)

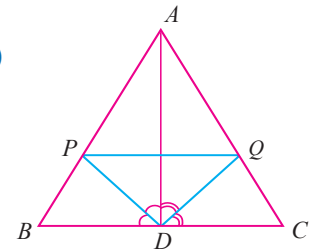
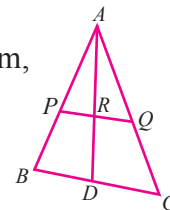


Fig. 6.15

Exercise 6.1

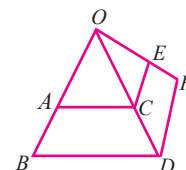
- In a $\triangle ABC$, D and E are points on the sides AB and AC respectively such that $DE \parallel BC$.
 - If $AD = 6$ cm, $DB = 9$ cm and $AE = 8$ cm, then find AC .
 - If $AD = 8$ cm, $AB = 12$ cm and $AE = 12$ cm, then find CE .
 - If $AD = 4x-3$, $BD = 3x-1$, $AE = 8x-7$ and $EC = 5x-3$, then find the value of x .

- In the figure, $AP = 3$ cm, $AR = 4.5$ cm, $AQ = 6$ cm, $AB = 5$ cm, and $AC = 10$ cm. Find the length of AD .



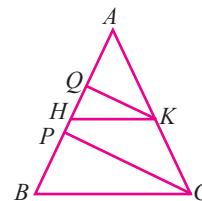
- E and F are points on the sides PQ and PR respectively, of a $\triangle PQR$. For each of the following cases, verify $EF \parallel QR$.
 - $PE = 3.9$ cm, $EQ = 3$ cm, $PF = 3.6$ cm and $FR = 2.4$ cm.
 - $PE = 4$ cm, $QE = 4.5$ cm, $PF = 8$ cm and $RF = 9$ cm.

- In the figure, $AC \parallel BD$ and $CE \parallel DF$. If $OA = 12$ cm, $AB = 9$ cm, $OC = 8$ cm and $EF = 4.5$ cm, then find FO .

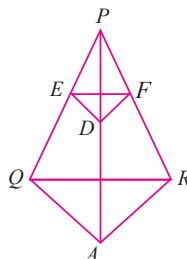


5. $ABCD$ is a quadrilateral with AB parallel to CD . A line drawn parallel to AB meets AD at P and BC at Q . Prove that $\frac{AP}{PD} = \frac{BQ}{QC}$.

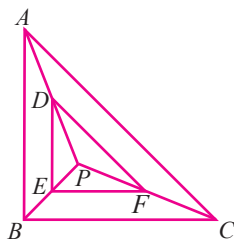
6. In the figure, $PC \parallel QK$ and $BC \parallel HK$. If $AQ = 6$ cm, $QH = 4$ cm, $HP = 5$ cm, $KC = 18$ cm, then find AK and PB .



7. In the figure, $DE \parallel AQ$ and $DF \parallel AR$. Prove that $EF \parallel QR$.



8. In the figure $DE \parallel AB$ and $DF \parallel AC$. Prove that $EF \parallel BC$.

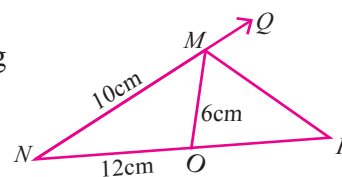


9. In a $\triangle ABC$, AD is the internal bisector of $\angle A$, meeting BC at D .
- If $BD = 2$ cm, $AB = 5$ cm, $DC = 3$ cm find AC .
 - If $AB = 5.6$ cm, $AC = 6$ cm and $DC = 3$ cm find BC .
 - If $AB = x$, $AC = x-2$, $BD = x+2$ and $DC = x-1$ find the value of x .

10. Check whether AD is the bisector of $\angle A$ of $\triangle ABC$ in each of the following.

- $AB = 4$ cm, $AC = 6$ cm, $BD = 1.6$ cm, and $CD = 2.4$ cm.
- $AB = 6$ cm, $AC = 8$ cm, $BD = 1.5$ cm and $CD = 3$ cm.

11. In a $\triangle MNO$, MP is the external bisector of $\angle M$ meeting NO produced at P . If $MN = 10$ cm, $MO = 6$ cm, $NO = 12$ cm, then find OP .



12. In a quadrilateral $ABCD$, the bisectors of $\angle B$ and $\angle D$ intersect on AC at E . Prove that $\frac{AB}{BC} = \frac{AD}{DC}$.

13. The internal bisector of $\angle A$ of $\triangle ABC$ meets BC at D and the external bisector of $\angle A$ meets BC produced at E . Prove that $\frac{BD}{BE} = \frac{CD}{CE}$.

14. $ABCD$ is a quadrilateral with $AB = AD$. If AE and AF are internal bisectors of $\angle BAC$ and $\angle DAC$ respectively, then prove that $EF \parallel BD$.

6.3 Similar triangles

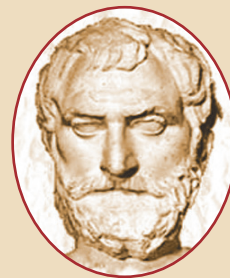
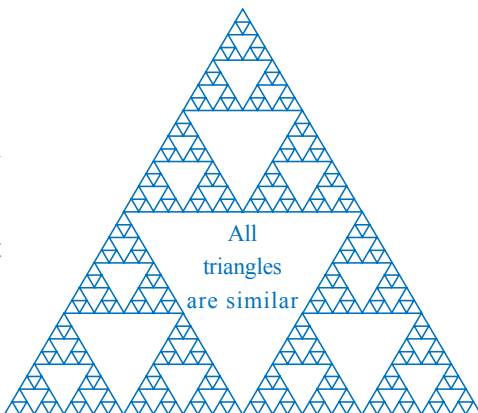
In class VIII, we have studied congruence of triangles in detail. We have learnt that two geometrical figures are congruent if they have the same size and shape. In this section, we shall study about those geometrical figures which have the same shape but not necessarily the same size. Such geometrical figures are called similar.

On looking around us, we see many objects which are of the same shape but of same or different sizes. For example, leaves of a tree have almost the same shapes but same or different sizes. Similarly photographs of different sizes developed from the same negative are of same shape but different sizes. All those objects which have the same shape but different sizes are called **similar objects**.

Thales said to have introduced Geometry in Greece, is believed to have found the heights of the Pyramids in Egypt, using shadows and the principle of similar triangles. Thus the use of similar triangles has made possible the measurements of heights and distances. He observed that the base angles of an isosceles triangle are equal. He used the idea of similar triangles and right triangles in practical geometry.

It is clear that the congruent figures are similar but the converse need not be true. In this section, we shall discuss

only the similarity of triangles and apply this knowledge in solving problems. The following simple activity helps us to visualize similar triangles.



Thales of Miletus

(624-546 BC)

Greece

Thales was the first known philosopher, scientist and mathematician. He is credited with the first use of deductive reasoning applied to geometry. He discovered many prepositions in geometry. His method of attacking problems invited the attention of many mathematicians. He also predicted an eclipse of the Sun in 585 BC.

Activity

- ❖ Take a cardboard and make a triangular hole in it.
- ❖ Expose this cardboard to Sunlight at about one metre above the ground .
- ❖ Move it towards the ground to see the formation of a sequence of triangular shapes on the ground.
- ❖ Moving close to the ground, the image becomes smaller and smaller. Moving away from the ground, the image becomes larger and larger.
- ❖ You see that, the size of the angles forming the three vertices of the triangle would always be the same, even though their sizes are different.

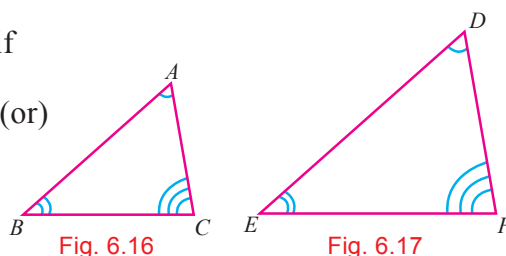
Definition

Two triangles are similar if

- (i) their corresponding angles are equal (or)
- (ii) their corresponding sides have lengths in the same ratio (or proportional), which is equivalent to saying that one triangle is an enlargement of other.

Thus, two triangles $\triangle ABC$ and $\triangle DEF$ are similar if

- (i) $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$ (or)
- (ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.



Here, the vertices A, B and C correspond to the vertices D, E and F respectively. Symbolically, we write the similarity of these two triangles as $\triangle ABC \sim \triangle DEF$ and read it as $\triangle ABC$ is similar to $\triangle DEF$. The symbol ' \sim ' stands for 'is similar to'.

Remarks

Similarity of $\triangle ABC$ and $\triangle DEF$ can also be expressed symbolically using correct correspondence of their vertices as $\triangle BCA \sim \triangle EFD$ and $\triangle CAB \sim \triangle FDE$.

6.3.1 Criteria for similarity of triangles

The following three criteria are sufficient to prove that two triangles are similar.

(i) AA (Angle-Angle) similarity criterion

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

Remark If two angles of a triangle are respectively equal to two angles of another triangle then their third angles will also be equal. Therefore AA similarity criterion is also referred to AAA criteria.

(ii) SSS (Side-Side-Side) similarity criterion for Two Triangles

In two triangles, if the sides of one triangle are proportional (in the same ratio) to the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similar.

(iii) SAS (Side-Angle-Side) similarity criterion for Two Triangles

If one angle of a triangle is equal to one angle of the other triangle and if the corresponding sides including these angles are proportional, then the two triangles are similar.

Let us list out a few results without proofs on similarity of triangles.

- (i) The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
- (ii) If a perpendicular is drawn from the vertex of a right angled triangle to its hypotenuse, then the triangles on each side of the perpendicular are similar to the whole triangle.

Here, (a) $\triangle DBA \sim \triangle ABC$

(b) $\triangle DAC \sim \triangle ABC$

(c) $\triangle DBA \sim \triangle DAC$

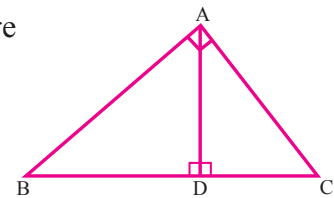


Fig. 6.18

- (iii) If two triangles are similar, then the ratio of the corresponding sides is equal to the ratio of their corresponding altitudes.

i.e., if $\triangle ABC \sim \triangle EFG$, then $\frac{AB}{EF} = \frac{BC}{FG} = \frac{CA}{GE} = \frac{AD}{EH}$

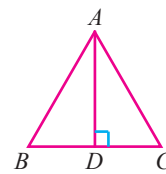


Fig. 6.19

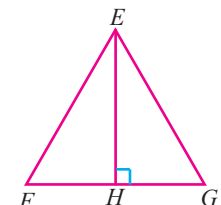


Fig. 6.20

- (iv) If two triangles are similar, then the ratio of the corresponding sides is equal to the ratio of the corresponding perimeters.

If, $\triangle ABC \sim \triangle DEF$, then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{AB + BC + CA}{DE + EF + FD}$.

Example 6.8

In $\triangle PQR$, $AB \parallel QR$. If AB is 3 cm, PB is 2 cm and PR is 6 cm, then find the length of QR .

Solution Given AB is 3 cm, PB is 2 cm PR is 6 cm and $AB \parallel QR$

In $\triangle PAB$ and $\triangle PQR$

$$\angle PAB = \angle PQR \quad (\text{corresponding angles})$$

and $\angle P$ is common.

$$\therefore \triangle PAB \sim \triangle PQR \quad (\text{AA similarity criterion})$$

Since corresponding sides are proportional,

$$\begin{aligned} \frac{AB}{QR} &= \frac{PB}{PR} \\ QR &= \frac{AB \times PR}{PB} \\ &= \frac{3 \times 6}{2} \end{aligned}$$

Thus, $QR = 9$ cm.

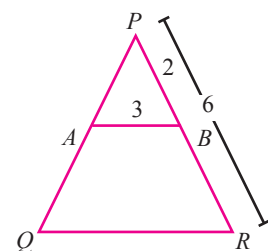


Fig. 6.21

Example 6.9

A man of height 1.8 m is standing near a Pyramid. If the shadow of the man is of length 2.7 m and the shadow of the Pyramid is 210 m long at that instant, find the height of the Pyramid.

Solution Let AB and DE be the heights of the Pyramid and the man respectively.

Let BC and EF be the lengths of the shadows of the Pyramid and the man respectively.

In $\triangle ABC$ and $\triangle DEF$, we have

$$\angle ABC = \angle DEF = 90^\circ$$

$$\angle BCA = \angle FED$$

(angular elevation is same at the same instant)

$$\therefore \triangle ABC \sim \triangle DEF \quad (\text{AA similarity criterion})$$

Thus,
$$\frac{AB}{DE} = \frac{BC}{EF}$$

$$\Rightarrow \frac{AB}{1.8} = \frac{210}{2.7} \Rightarrow AB = \frac{210}{2.7} \times 1.8 = 140.$$

Hence, the height of the Pyramid is 140 m.

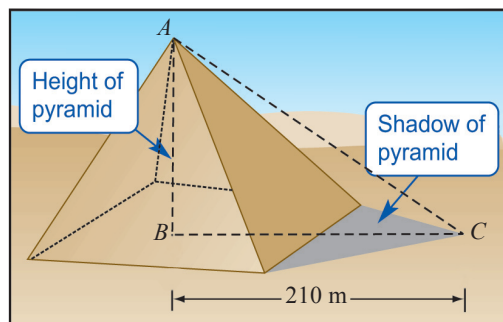


Fig. 6.22

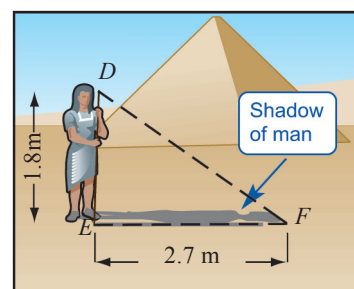


Fig. 6.23

Example 6.10

A man sees the top of a tower in a mirror which is at a distance of 87.6 m from the tower. The mirror is on the ground, facing upward. The man is 0.4 m away from the mirror, and the distance of his eye level from the ground is 1.5 m. How tall is the tower? (The foot of man, the mirror and the foot of the tower lie along a straight line.)

Solution Let AB and ED be the heights of the man and the tower respectively. Let C be the point of incidence of the tower in the mirror.

In $\triangle ABC$ and $\triangle EDC$, we have

$$\angle ABC = \angle EDC = 90^\circ$$

$$\angle BCA = \angle DCE$$

(angular elevation is same at the same instant. i.e., the angle of incidence and the angle of reflection are same.)

$$\therefore \triangle ABC \sim \triangle EDC \quad (\text{AA similarity criterion})$$

Thus,
$$\frac{ED}{AB} = \frac{DC}{BC} \quad (\text{corresponding sides are proportional})$$

$$ED = \frac{DC}{BC} \times AB = \frac{87.6}{0.4} \times 1.5 = 328.5$$

Hence, the height of the tower is 328.5 m.

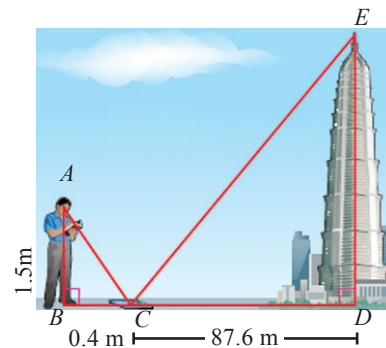


Fig. 6.24

Example 6.11

The image of a tree on the film of a camera is of length 35 mm, the distance from the lens to the film is 42 mm and the distance from the lens to the tree is 6 m. How tall is the portion of the tree being photographed?

Solution Let AB and EF be the heights of the portion of the tree and its image on the film respectively.

Let the point C denote the lens.

Let CG and CH be altitudes of $\triangle ACB$ and $\triangle FEC$.

Clearly, $AB \parallel FE$.

In $\triangle ACB$ and $\triangle FEC$,

$$\angle BAC = \angle FEC$$

$$\angle ECF = \angle ACB \text{ (vertically opposite angles)}$$

$\therefore \triangle ACB \sim \triangle FEC$ (AA criterion)

$$\text{Thus, } \frac{AB}{EF} = \frac{CG}{CH}$$

$$\Rightarrow AB = \frac{CG}{CH} \times EF = \frac{6 \times 0.035}{0.042} = 5.$$

Hence, the height of the tree photographed is 5m.

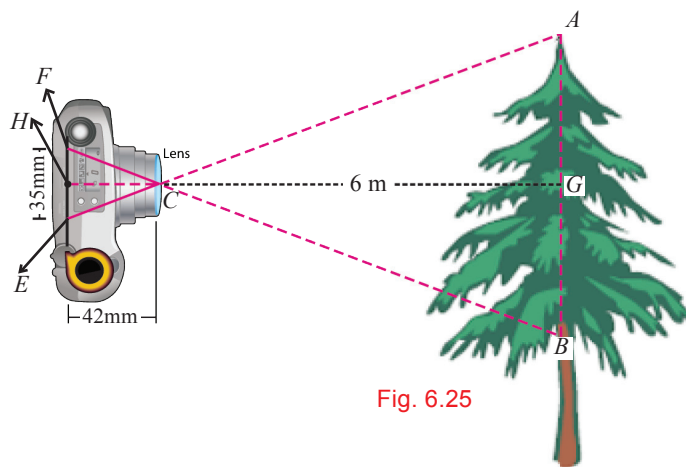
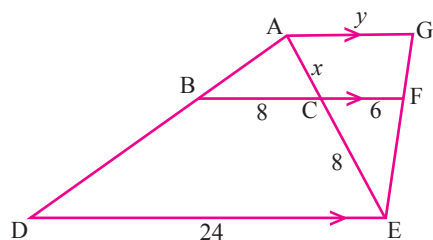


Fig. 6.25

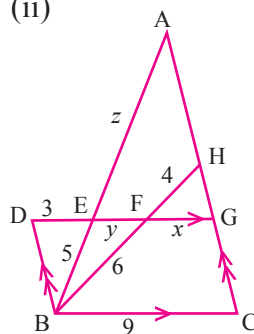
Exercise 6.2

- Find the unknown values in each of the following figures. All lengths are given in centimetres. (All measures are not in scale)

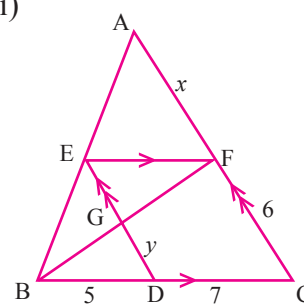
(i)



(ii)

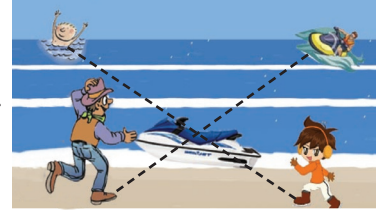


(iii)



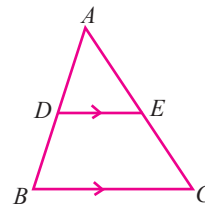
- The image of a man of height 1.8 m, is of length 1.5 cm on the film of a camera. If the film is 3 cm from the lens of the camera, how far is the man from the camera?
- A girl of height 120 cm is walking away from the base of a lamp-post at a speed of 0.6 m/sec. If the lamp is 3.6 m above the ground level, then find the length of her shadow after 4 seconds.

4. A girl is in the beach with her father. She spots a swimmer drowning. She shouts to her father who is 50 m due west of her. Her father is 10 m nearer to a boat than the girl. If her father uses the boat to reach the swimmer, he has to travel a distance 126 m from that boat. At the same time, the girl spots a man riding a water craft who is 98 m away from the boat. The man on the water craft is due east of the swimmer. How far must the man travel to rescue the swimmer? (Hint : see figure). (Not for the examination)

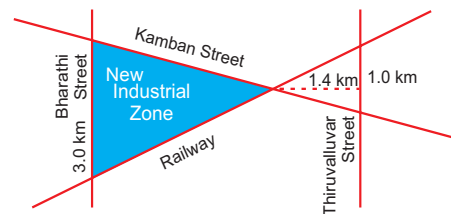


5. P and Q are points on sides AB and AC respectively, of $\triangle ABC$. If $AP = 3$ cm, $PB = 6$ cm, $AQ = 5$ cm and $QC = 10$ cm, show that $BC = 3 PQ$.
6. In $\triangle ABC$, $AB = AC$ and $BC = 6$ cm. D is a point on the side AC such that $AD = 5$ cm and $CD = 4$ cm. Show that $\triangle BCD \sim \triangle ACB$ and hence find BD .
7. The points D and E are on the sides AB and AC of $\triangle ABC$ respectively, such that $DE \parallel BC$. If $AB = 3 AD$ and the area of $\triangle ABC$ is 72 cm^2 , then find the area of the quadrilateral $DBCE$.
8. The lengths of three sides of a triangle ABC are 6 cm, 4 cm and 9 cm. $\triangle PQR \sim \triangle ABC$. One of the lengths of sides of $\triangle PQR$ is 35 cm. What is the greatest perimeter possible for $\triangle PQR$?

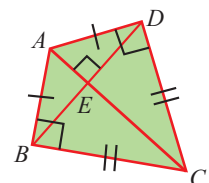
9. In the figure, $DE \parallel BC$ and $\frac{AD}{BD} = \frac{3}{5}$, calculate the value of
- (i) $\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}$, (ii) $\frac{\text{area of trapezium } BCED}{\text{area of } \triangle ABC}$



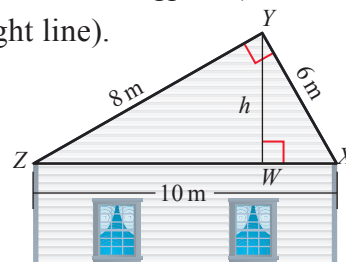
10. The government plans to develop a new industrial zone in an unused portion of land in a city. The shaded portion of the map shown on the right, indicates the area of the new industrial zone. Find the area of the new industrial zone.



11. A boy is designing a diamond shaped kite, as shown in the figure where $AE = 16$ cm, $EC = 81$ cm. He wants to use a straight cross bar BD . How long should it be?



12. A student wants to determine the height of a flagpole. He placed a small mirror on the ground so that he can see the reflection of the top of the flagpole. The distance of the mirror from him is 0.5 m and the distance of the flagpole from the mirror is 3 m. If his eyes are 1.5 m above the ground level, then find the height of the flagpole. (The foot of student, mirror and the foot of flagpole lie along a straight line).
13. A roof has a cross section as shown in the diagram,
 (i) Identify the similar triangles
 (ii) Find the height h of the roof.



Theorem 6.5

Pythagoras theorem (Baudhayan theorem)

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Given : In a right angled $\triangle ABC$, $\angle A = 90^\circ$.

To prove : $BC^2 = AB^2 + AC^2$

Construction : Draw $AD \perp BC$

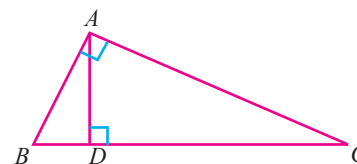


Fig. 6.26

Proof

In triangles ABC and DBA , $\angle B$ is the common angle.

Also, we have $\angle BAC = \angle ADB = 90^\circ$.

$\therefore \triangle ABC \sim \triangle DBA$ (AA similarity criterion)

Thus, their corresponding sides are proportional.

$$\begin{aligned} \text{Hence, } \frac{AB}{DB} &= \frac{BC}{BA} \\ \therefore AB^2 &= DB \times BC \end{aligned} \quad (1)$$

Similarly, we have $\triangle ABC \sim \triangle DAC$.

$$\begin{aligned} \text{Thus, } \frac{BC}{AC} &= \frac{AC}{DC} \\ \therefore AC^2 &= BC \times DC \end{aligned} \quad (2)$$

Adding (1) and (2) we get,

$$\begin{aligned} AB^2 + AC^2 &= BD \times BC + BC \times DC \\ &= BC(BD + DC) \\ &= BC \times BC = BC^2 \end{aligned}$$

Thus, $BC^2 = AB^2 + AC^2$. Hence the Pythagoras theorem.

Remarks

The Pythagoras theorem has two fundamental aspects; one is about areas and the other is about lengths. Hence this landmark theorem connects Geometry and Algebra. The converse of Pythagoras theorem is also true. It was first mentioned and proved by **Euclid**.

The statement is given below. (Proof is left as an exercise.)

Theorem 6.6 Converse of Pythagoras theorem

In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

6.4 Circles and Tangents

A straight line associated with circles is a tangent line which touches the circle at just one point. In geometry, tangent lines to circles play an important role in many geometrical constructions and proofs. In this section, let us state some results based on circles and tangents and prove an important theorem known as Tangent-Chord theorem. If we consider a straight line and a circle in a plane, then there are three possibilities- they may not intersect at all, they may intersect at two points or they may touch each other at exactly one point. Now look at the following figures.

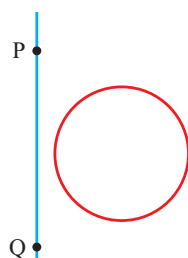


Fig. 6.27

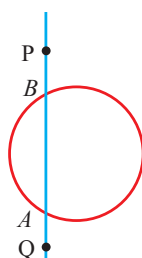


Fig. 6.28

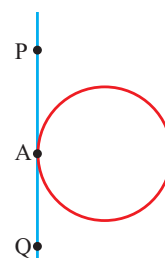


Fig. 6.29

In Fig. 6.27, the circle and the straight line PQ have no common point.

In Fig. 6.28, the straight line PQ cuts the circle at two distinct points A and B . In this case, PQ is called a **secant** to the circle.

In Fig. 6.29, the straight line PQ and the circle have exactly one common point. Equivalently the straight line touches the circle at only one point. The straight line PQ is called the **tangent** to the circle at A .

Definition

A straight line which touches a circle at only one point is called a **tangent** to the circle and the point at which it touches the circle is called its point of contact.

Theorems based on circles and tangents (without proofs)

1. A tangent at any point on a circle is perpendicular to the radius through the point of contact .
2. Only one tangent can be drawn at any point on a circle. However, from an exterior point of a circle two tangents can be drawn to the circle.
3. The lengths of the two tangents drawn from an exterior point to a circle are equal.
4. If two circles touch each other, then the point of contact of the circles lies on the line joining the centres.
5. If two circles touch externally, the distance between their centres is equal to the sum of their radii.
6. If two circles touch internally, the distance between their centres is equal to the difference of their radii.

Theorem 6.7 Tangent-Chord theorem

If from the point of contact of tangent (of a circle), a chord is drawn, then the angles which the chord makes with the tangent line are equal respectively to the angles formed by the chord in the corresponding alternate segments.

Given : O is the centre of the circle. ST is the tangent at A , and AB is a chord. P and Q are any two points on the circle in the opposite sides of the chord AB .

To prove : (i) $\angle BAT = \angle BPA$ (ii) $\angle BAS = \angle AQB$.

Construction: Draw the diameter AC of the circle. Join B and C .

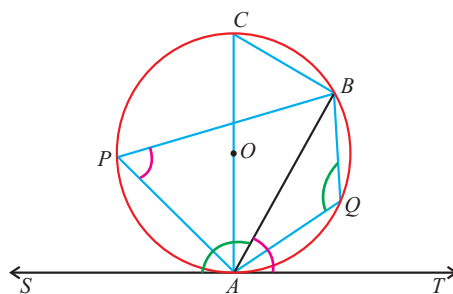


Fig. 6.30

Proof

Statement

$$\angle ABC = 90^\circ$$

$$\angle CAB + \angle BCA = 90^\circ$$

$$\angle CAT = 90^\circ$$

$$\Rightarrow \angle CAB + \angle BAT = 90^\circ$$

$$\angle CAB + \angle BCA = \angle CAB + \angle BAT$$

$$\Rightarrow \angle BCA = \angle BAT$$

Reason

angle in a semi-circle is 90°

sum of two acute angles of a right $\triangle ABC$. (1)

diameter is \perp to the tangent at the point of contact.

(2)

from (1) and (2)

(3)

$$\angle BCA = \angle BPA \quad \text{angles in the same segment} \quad (4)$$

$$\angle BAT = \angle BPA \quad \text{Hence (i).} \quad \text{from (3) and (4)} \quad (5)$$

$$\text{Now } \angle BPA + \angle AQB = 180^\circ \quad \text{opposite angles of a cyclic quadrilateral}$$

$$\Rightarrow \angle BAT + \angle AQB = 180^\circ \quad \text{from (5)} \quad (6)$$

$$\text{Also } \angle BAT + \angle BAS = 180^\circ \quad \text{linear pair} \quad (7)$$

$$\angle BAT + \angle AQB = \angle BAT + \angle BAS \quad \text{from (6) and (7)}$$

$$\angle BAS = \angle AQB. \quad \text{Hence (ii).}$$

Thus, the Tangent -Chord theorem is proved.

Theorem 6.8

Converse of Tangent-Chord theorem

If in a circle, through one end of a chord, a straight line is drawn making an angle equal to the angle in the alternate segment, then the straight line is a tangent to the circle.

Definition

Let P be a point on a line segment AB . The product $PA \times PB$ represents the area of the rectangle whose sides are PA and PB .

This product is called the area of the rectangle contained by the parts PA and PB of the line segment AB .



Theorem 6.9

If two chords of a circle intersect either inside or outside of the circle, then the area of the rectangle contained by the segments of one chord is equal to the area of the rectangle contained by the segments of the other chord.

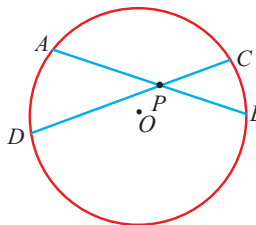


Fig. 6.31

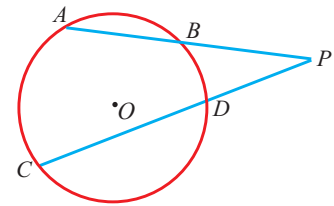


Fig. 6.32

In Fig.6.31, two chords AB and CD intersect at P inside the circle with centre at O . Then $PA \times PB = PC \times PD$. In Fig. 6.32, the chords AB and CD intersect at P outside the circle with centre O . Then $PA \times PB = PC \times PD$.

Example 6.12

Let PQ be a tangent to a circle at A and AB be a chord. Let C be a point on the circle such that $\angle BAC = 54^\circ$ and $\angle BAQ = 62^\circ$. Find $\angle ABC$.

Solution Since PQ is a tangent at A and AB is a chord, we have

$$\angle BAQ = \angle ACB = 62^\circ. \quad (\text{tangent-chord theorem})$$

Also, $\angle BAC + \angle ACB + \angle ABC = 180^\circ$.

(sum of all angles in a triangle is 180°)

Thus, $\angle ABC = 180^\circ - (\angle BAC + \angle ACB)$

Hence, $\angle ABC = 180^\circ - (54^\circ + 62^\circ) = 64^\circ$.

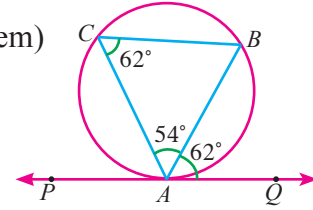


Fig. 6.33

Example 6.13

Find the value of x in each of the following diagrams.

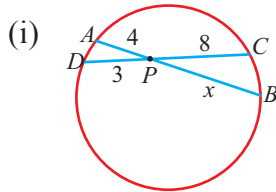


Fig. 6.34

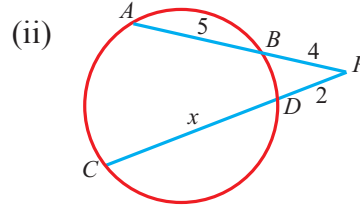


Fig. 6.35

Solution (i) We have $PA \cdot PB = PC \cdot PD$

$$PB = \frac{PC \cdot PD}{PA}$$

Thus, $x = \frac{8 \times 3}{4} = 6$.

(ii) We have $PC \cdot PD = PA \cdot PB$

$$(2+x) \cdot 2 = 9 \times 4$$

$$x + 2 = 18. \quad \text{Thus, } x = 16.$$

Example 6.14

In the figure, tangents PA and PB are drawn to a circle with centre O from an external point P . If CD is a tangent to the circle at E and $AP = 15$ cm, find the perimeter of $\triangle PCD$

Solution We know that the lengths of the two tangents from an exterior point to a circle are equal.

$$\therefore CA = CE, \quad DB = DE \text{ and } PA = PB.$$

Now, the perimeter of $\triangle PCD = PC + CD + DP$

$$= PC + CE + ED + DP$$

$$= PC + CA + DB + DP$$

$$= PA + PB = 2PA \quad (PB = PA)$$

Thus, the perimeter of $\triangle PCD = 2 \times 15 = 30$ cm.

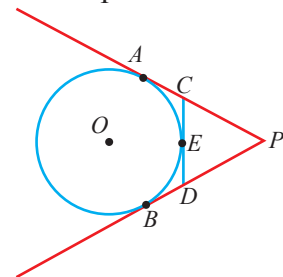


Fig. 6.36

Example 6.15

$ABCD$ is a quadrilateral such that all of its sides touch a circle. If $AB = 6$ cm, $BC = 6.5$ cm and $CD = 7$ cm, then find the length of AD .

Solution Let P, Q, R and S be the points where the circle touches the quadrilateral. We know that the lengths of the two tangents drawn from an exterior point to a circle are equal. Thus, we have, $AP = AS, BP = BQ, CR = CQ$ and $DR = DS$.

Hence, $AP + BP + CR + DR = AS + BQ + CQ + DS$

$$\Rightarrow AB + CD = AD + BC.$$

$$\begin{aligned} \Rightarrow AD &= AB + CD - BC \\ &= 6 + 7 - 6.5 = 6.5 \end{aligned}$$

Thus, $AD = 6.5$ cm.

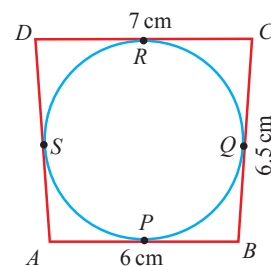
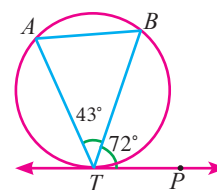


Fig. 6.37

Exercise 6.3

- In the figure TP is a tangent to a circle. A and B are two points on the circle. If $\angle BTP = 72^\circ$ and $\angle ATB = 43^\circ$ find $\angle ABT$.
- AB and CD are two chords of a circle which intersect each other internally at P . (i) If $CP = 4$ cm, $AP = 8$ cm, $PB = 2$ cm, then find PD .
(ii) If $AP = 12$ cm, $AB = 15$ cm, $CP = PD$, then find CD .
- AB and CD are two chords of a circle which intersect each other externally at P .
(i) If $AB = 4$ cm, $BP = 5$ cm and $PD = 3$ cm, then find CD .
(ii) If $BP = 3$ cm, $CP = 6$ cm and $CD = 2$ cm, then find AB .
- A circle touches the side BC of $\triangle ABC$ at P , AB and AC produced at Q and R respectively, prove that $AQ = AR = \frac{1}{2}$ (perimeter of $\triangle ABC$)
- If all sides of a parallelogram touch a circle, show that the parallelogram is a rhombus.
- A lotus is 20 cm above the water surface in a pond and its stem is partly below the water surface. As the wind blew, the stem is pushed aside so that the lotus touched the water 40 cm away from the original position of the stem. How much of the stem was below the water surface originally?
- A point O in the interior of a rectangle $ABCD$ is joined to each of the vertices A, B, C and D . Prove that $OA^2 + OC^2 = OB^2 + OD^2$



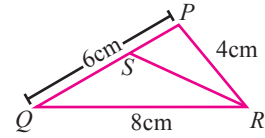
Exercise 6.4

Choose the correct answer

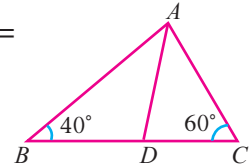
- If a straight line intersects the sides AB and AC of a $\triangle ABC$ at D and E respectively and is parallel to BC , then $\frac{AE}{AC} =$
 (A) $\frac{AD}{DB}$ (B) $\frac{AD}{AB}$ (C) $\frac{DE}{BC}$ (D) $\frac{AD}{EC}$

2. In $\triangle ABC$, DE is \parallel to BC , meeting AB and AC at D and E .
If $AD = 3$ cm, $DB = 2$ cm and $AE = 2.7$ cm, then AC is equal to
(A) 6.5 cm (B) 4.5 cm (C) 3.5 cm (D) 5.5 cm

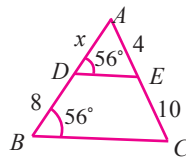
3. In $\triangle PQR$, RS is the bisector of $\angle R$. If $PQ = 6$ cm, $QR = 8$ cm, $RP = 4$ cm then PS is equal to
(A) 2 cm (B) 4 cm (C) 3 cm (D) 6 cm



4. In figure, if $\frac{AB}{AC} = \frac{BD}{DC}$, $\angle B = 40^\circ$, and $\angle C = 60^\circ$, then $\angle BAD =$
(A) 30° (B) 50° (C) 80° (D) 40°



5. In the figure, the value x is equal to
(A) $4 \cdot 2$ (B) $3 \cdot 2$
(C) $0 \cdot 8$ (D) $0 \cdot 4$

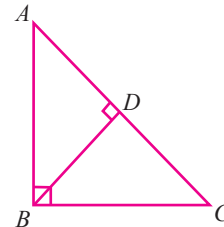


6. In triangles ABC and DEF , $\angle B = \angle E$, $\angle C = \angle F$, then

(A) $\frac{AB}{DE} = \frac{CA}{EF}$ (B) $\frac{BC}{EF} = \frac{AB}{FD}$ (C) $\frac{AB}{DE} = \frac{BC}{EF}$ (D) $\frac{CA}{FD} = \frac{AB}{EF}$

7. From the given figure, identify the wrong statement.

- (A) $\triangle ADB \sim \triangle ABC$ (B) $\triangle ABD \sim \triangle ABC$
(C) $\triangle BDC \sim \triangle ABC$ (D) $\triangle ADB \sim \triangle BDC$



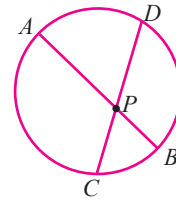
8. If a vertical stick 12 m long casts a shadow 8 m long on the ground and at the same time a tower casts a shadow 40 m long on the ground, then the height of the tower is
(A) 40 m (B) 50 m (C) 75 m (D) 60 m
9. The sides of two similar triangles are in the ratio 2:3, then their areas are in the ratio
(A) 9:4 (B) 4:9 (C) 2:3 (D) 3:2
10. Triangles ABC and DEF are similar. If their areas are 100cm^2 and 49cm^2 respectively and BC is 8.2 cm then $EF =$
(A) 5.47 cm (B) 5.74 cm (C) 6.47 cm (D) 6.74 cm
11. The perimeters of two similar triangles are 24 cm and 18 cm respectively. If one side of the first triangle is 8 cm, then the corresponding side of the other triangle is
(A) 4 cm (B) 3 cm (C) 9 cm (D) 6 cm

12. AB and CD are two chords of a circle which when produced to meet at a point P such that $AB = 5$ cm, $AP = 8$ cm, and $CD = 2$ cm then $PD =$

(A) 12 cm (B) 5 cm (C) 6 cm (D) 4 cm

13. In the adjoining figure, chords AB and CD intersect at P . If $AB = 16$ cm, $PD = 8$ cm, $PC = 6$ and $AP > PB$, then $AP =$

(A) 8 cm (B) 4 cm (C) 12 cm (D) 6 cm

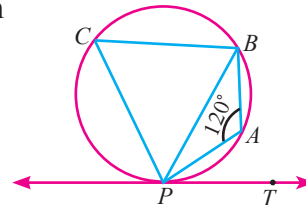


14. A point P is 26 cm away from the centre O of a circle and PT is the tangent drawn from P to the circle is 10 cm, then OT is equal to

(A) 36 cm (B) 20 cm (C) 18 cm (D) 24 cm

15. In the figure, if $\angle PAB = 120^\circ$ then $\angle BPT =$

(A) 120° (B) 30° (C) 40° (D) 60°

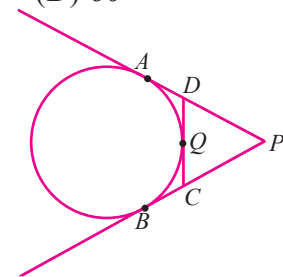


16. If the tangents PA and PB from an external point P to circle with centre O are inclined to each other at an angle of 40° , then $\angle POA =$

(A) 70° (B) 80° (C) 50° (D) 60°

17. In the figure, PA and PB are tangents to the circle drawn from an external point P . Also CD is a tangent to the circle at Q . If $PA = 8$ cm and $CQ = 3$ cm, then PC is equal to

(A) 11 cm (B) 5 cm (C) 24 cm (D) 38 cm



18. $\triangle ABC$ is a right angled triangle where $\angle B = 90^\circ$ and $BD \perp AC$. If $BD = 8$ cm, $AD = 4$ cm, then CD is

(A) 24 cm (B) 16 cm (C) 32 cm (D) 8 cm

19. The areas of two similar triangles are 16 cm^2 and 36 cm^2 respectively. If the altitude of the first triangle is 3 cm, then the corresponding altitude of the other triangle is

(A) 6.5 cm (B) 6 cm (C) 4 cm (D) 4.5 cm

20. The perimeter of two similar triangles $\triangle ABC$ and $\triangle DEF$ are 36 cm and 24 cm respectively. If $DE = 10$ cm, then AB is

(A) 12 cm (B) 20 cm (C) 15 cm (D) 18 cm

7

- Introduction
- Identities
- Heights and Distances



Hipparchus
(190 - 120 B.C.)
Greece

Hipparchus developed trigonometry, constructed trigonometric tables and solved several problems of spherical trigonometry. With his solar and lunar theories and his trigonometry, he may have been the first to develop a reliable method to predict solar eclipses.

Hipparchus is credited with the invention or improvement of several astronomical instruments, which were used for a long time for naked-eye observations.

TRIGONOMETRY

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry – J.F. Herbart

7.1 Introduction

Trigonometry was developed to express relationship between the sizes of arcs in circles and the chords determining those arcs. After 15th century it was used to relate the measure of angles in a triangle to the lengths of the sides of the triangle. The creator of Trigonometry is said to have been the Greek Hipparchus of the second century B.C. The word **Trigonometry** which means triangle measurement, is credited to Bartholomaeus Pitiscus (1561-1613).

We have learnt in class IX about various trigonometric ratios, relation between them and how to use trigonometric tables in solving problems.

In this chapter, we shall learn about trigonometric identities, application of trigonometric ratios in finding heights and distances of hills, buildings etc., without actually measuring them.

7.2 Trigonometric identities

We know that an equation is called an **identity** when it is true for all values of the variable(s) for which the equation is meaningful. For example, the equation $(a + b)^2 = a^2 + 2ab + b^2$ is an identity since it is true for all real values of a and b .

Likewise, an equation involving trigonometric ratios of an angle is called a **trigonometric identity**, if it is true for all values of the angle(s) involved in the equation. For example, the equation $(\sin \theta + \cos \theta)^2 - (\sin \theta - \cos \theta)^2 = 4 \sin \theta \cos \theta$ is a trigonometric identity as it is true for all values of θ .

However, the equation $(\sin \theta + \cos \theta)^2 = 1$ is not an identity because it is true when $\theta = 0^\circ$, but not true when $\theta = 45^\circ$ as $(\sin 45^\circ + \cos 45^\circ)^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 = 2 \neq 1$.

In this chapter, all the trigonometric identities and equations are assumed to be well defined for those values of the variables for which they are meaningful.

Let us establish three useful identities called the **Pythagorean identities** and use them to obtain some other identities.

In the right-angled $\triangle ABC$, we have

$$AB^2 + BC^2 = AC^2 \quad (1)$$

Dividing each term of (1) by AC^2 , we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2} \quad (AC \neq 0)$$

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

Thus, $\cos^2 A + \sin^2 A = 1$

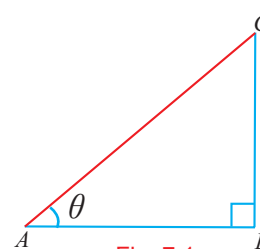


Fig. 7.1

Let $\angle A = \theta$. Then for all $0^\circ < \theta < 90^\circ$ we have,

$$\cos^2 \theta + \sin^2 \theta = 1. \quad (2)$$

Evidently, $\cos^2 0^\circ + \sin^2 0^\circ = 1$ and $\cos^2 90^\circ + \sin^2 90^\circ = 1$ and so (2) is true for all θ such that $0^\circ \leq \theta \leq 90^\circ$

Let us divide (1) by AB^2 , we get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \left(\frac{AC}{AB}\right)^2 \quad (\because AB \neq 0)$$

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2 \implies 1 + \tan^2 \theta = \sec^2 \theta. \quad (3)$$

Since $\tan \theta$ and $\sec \theta$ are not defined for $\theta = 90^\circ$, the identity (3) is true for all θ such that $0^\circ \leq \theta < 90^\circ$

Again dividing each term of (1) by BC^2 , we get

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \left(\frac{AC}{BC}\right)^2 \quad (\because BC \neq 0)$$

$$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2 \implies \cot^2 \theta + 1 = \operatorname{cosec}^2 \theta. \quad (4)$$

Since $\cot \theta$ and $\operatorname{cosec} \theta$ are not defined for $\theta = 0^\circ$, the identity (4) is true for all θ such that $0^\circ < \theta \leq 90^\circ$

Some equal forms of identities from (2) to (4) are listed below.

	Identity	Equal forms
(i)	$\sin^2 \theta + \cos^2 \theta = 1$	$\sin^2 \theta = 1 - \cos^2 \theta$ (or) $\cos^2 \theta = 1 - \sin^2 \theta$
(ii)	$1 + \tan^2 \theta = \sec^2 \theta$	$\sec^2 \theta - \tan^2 \theta = 1$ (or) $\tan^2 \theta = \sec^2 \theta - 1$
(iii)	$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$	$\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$ (or) $\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$

Remarks

We have proved the above identities for an acute angle θ . But these identities are true for any angle θ for which the trigonometric functions are meaningful. In this book we shall restrict ourselves to acute angles only.

In general, there is no common method for proving trigonometric identities involving trigonometric functions. However, some of the techniques listed below may be useful in proving trigonometric identities.

- Study the identity carefully, keeping in mind what is given and what you need to arrive.
- Generally, the more complicated side of the identity may be taken first and simplified as it is easier to simplify than to expand or enlarge the simpler one.
- If both sides of the identity are complicated, each may be taken individually and simplified independently of each other to the same expression.
- Combine fractions using algebraic techniques for adding expressions.
- If necessary, change each term into their sine and cosine equivalents and then try to simplify.
- If an identity contains terms involving $\tan^2 \theta, \cot^2 \theta, \operatorname{cosec}^2 \theta, \sec^2 \theta$, it may be more helpful to use the results $\sec^2 \theta = 1 + \tan^2 \theta$ and $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$.

Example 7.1

Prove the identity $\frac{\sin \theta}{\operatorname{cosec} \theta} + \frac{\cos \theta}{\sec \theta} = 1$

Solution

$$\begin{aligned} \text{Now, } \frac{\sin \theta}{\operatorname{cosec} \theta} + \frac{\cos \theta}{\sec \theta} &= \frac{\sin \theta}{\left(\frac{1}{\sin \theta}\right)} + \frac{\cos \theta}{\left(\frac{1}{\cos \theta}\right)} \\ &= \sin^2 \theta + \cos^2 \theta = 1. \end{aligned}$$

Example 7.2

Prove the identity $\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \operatorname{cosec} \theta - \cot \theta$

Solution

$$\begin{aligned}\text{Consider } \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} &= \sqrt{\frac{(1 - \cos \theta)}{(1 + \cos \theta)} \times \frac{(1 - \cos \theta)}{(1 - \cos \theta)}} \\ &= \sqrt{\frac{(1 - \cos \theta)^2}{1^2 - \cos^2 \theta}} = \sqrt{\frac{(1 - \cos \theta)^2}{\sin^2 \theta}} \quad (1 - \cos^2 \theta = \sin^2 \theta) \\ &= \frac{1 - \cos \theta}{\sin \theta} = \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \\ &= \operatorname{cosec} \theta - \cot \theta.\end{aligned}$$

Example 7.3

Prove the identity $[\operatorname{cosec}(90^\circ - \theta) - \sin(90^\circ - \theta)][\operatorname{cosec} \theta - \sin \theta][\tan \theta + \cot \theta] = 1$

Solution

$$\begin{aligned}\text{Now, } &[\operatorname{cosec}(90^\circ - \theta) - \sin(90^\circ - \theta)][\operatorname{cosec} \theta - \sin \theta][\tan \theta + \cot \theta] \\ &= (\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta) \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right) \quad \because \operatorname{cosec}(90^\circ - \theta) = \sec \theta \\ &\quad \because \sin(90^\circ - \theta) = \cos \theta \\ &= \left(\frac{1}{\cos \theta} - \cos \theta \right) \left(\frac{1}{\sin \theta} - \sin \theta \right) \left(\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \right) \\ &= \left(\frac{1 - \cos^2 \theta}{\cos \theta} \right) \left(\frac{1 - \sin^2 \theta}{\sin \theta} \right) \left(\frac{1}{\sin \theta \cos \theta} \right) \\ &= \left(\frac{\sin^2 \theta}{\cos \theta} \right) \left(\frac{\cos^2 \theta}{\sin \theta} \right) \left(\frac{1}{\sin \theta \cos \theta} \right) = 1\end{aligned}$$

Example 7.4

Prove that $\frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \frac{1 + \sin \theta}{\cos \theta}$

Solution

$$\begin{aligned}\text{We consider } &\frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} \\ &= \frac{\tan \theta + \sec \theta - (\sec^2 \theta - \tan^2 \theta)}{\tan \theta - \sec \theta + 1} \quad (\sec^2 \theta - \tan^2 \theta = 1) \\ &= \frac{(\tan \theta + \sec \theta) - (\sec \theta + \tan \theta)(\sec \theta - \tan \theta)}{\tan \theta - \sec \theta + 1} \quad (a^2 - b^2 = (a + b)(a - b)) \\ &= \frac{(\tan \theta + \sec \theta)[1 - (\sec \theta - \tan \theta)]}{\tan \theta - \sec \theta + 1} \\ &= \frac{(\tan \theta + \sec \theta)(\tan \theta - \sec \theta + 1)}{\tan \theta - \sec \theta + 1} \\ &= \tan \theta + \sec \theta = \frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} = \frac{1 + \sin \theta}{\cos \theta}\end{aligned}$$

Example 7.5

Prove the identity $\frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta} = 1 + \tan \theta + \cot \theta$.

Solution

$$\begin{aligned} \text{Now, } & \frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta} \\ &= \frac{\tan \theta}{1 - \frac{1}{\tan \theta}} + \frac{\frac{1}{\tan \theta}}{1 - \tan \theta} = \frac{\tan \theta}{\frac{\tan \theta - 1}{\tan \theta}} + \frac{\frac{1}{\tan \theta}}{1 - \tan \theta} \\ &= \frac{\tan^2 \theta}{\tan \theta - 1} + \frac{1}{\tan \theta(1 - \tan \theta)} = \frac{\tan^2 \theta}{\tan \theta - 1} + \frac{1}{(-\tan \theta)(\tan \theta - 1)} \\ &= \frac{\tan^2 \theta}{\tan \theta - 1} - \frac{1}{(\tan \theta)(\tan \theta - 1)} \\ &= \frac{1}{(\tan \theta - 1)} \left(\tan^2 \theta - \frac{1}{\tan \theta} \right) \\ &= \frac{1}{(\tan \theta - 1)} \frac{(\tan^3 \theta - 1)}{\tan \theta} \\ &= \frac{(\tan \theta - 1)(\tan^2 \theta + \tan \theta + 1^2)}{(\tan \theta - 1)\tan \theta} \quad (\because a^3 - b^3 = (a - b)(a^2 + ab + b^2)) \\ &= \frac{\tan^2 \theta + \tan \theta + 1}{\tan \theta} \\ &= \frac{\tan^2 \theta}{\tan \theta} + \frac{\tan \theta}{\tan \theta} + \frac{1}{\tan \theta} = \tan \theta + 1 + \cot \theta \\ &= 1 + \tan \theta + \cot \theta. \end{aligned}$$

Example 7.6

Prove the identity

$$(\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2 = 7 + \tan^2 \theta + \cot^2 \theta.$$

Solution

$$\begin{aligned} \text{Let us consider } & (\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2 \\ &= \sin^2 \theta + \operatorname{cosec}^2 \theta + 2 \sin \theta \operatorname{cosec} \theta + \cos^2 \theta + \sec^2 \theta + 2 \cos \theta \sec \theta \\ &= \sin^2 \theta + \cos^2 \theta + \operatorname{cosec}^2 \theta + \sec^2 \theta + 2 \sin \theta \frac{1}{\sin \theta} + 2 \cos \theta \frac{1}{\cos \theta} \\ &= 1 + (1 + \cot^2 \theta) + (1 + \tan^2 \theta) + 2 + 2 \\ &= 7 + \tan^2 \theta + \cot^2 \theta. \end{aligned}$$

Example 7.7

Prove the identity $(\sin^6 \theta + \cos^6 \theta) = 1 - 3 \sin^2 \theta \cos^2 \theta$.

Solution

$$\begin{aligned}\text{Now } \sin^6 \theta + \cos^6 \theta &= (\sin^2 \theta)^3 + (\cos^2 \theta)^3 \\ &= (\sin^2 \theta + \cos^2 \theta)^3 - 3 \sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta) \\ &= 1 - 3 \sin^2 \theta \cos^2 \theta. \quad \begin{array}{l} (a^3 + b^3 = (a + b)^3 - 3ab(a + b)) \\ (\sin^2 \theta + \cos^2 \theta = 1) \end{array}\end{aligned}$$

Example 7.8

Prove the identity $\frac{\sin \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - \cos \theta} = \tan \theta$.

Solution

$$\begin{aligned}\text{Now, } \frac{\sin \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - \cos \theta} &= \frac{\sin \theta (1 - 2 \sin^2 \theta)}{\cos \theta (2 \cos^2 \theta - 1)} \\ &= \left(\frac{\sin \theta}{\cos \theta} \right) \left(\frac{\sin^2 \theta + \cos^2 \theta - 2 \sin^2 \theta}{2 \cos^2 \theta - (\sin^2 \theta + \cos^2 \theta)} \right) \quad (\sin^2 \theta + \cos^2 \theta = 1) \\ &= (\tan \theta) \left(\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta} \right) = \tan \theta.\end{aligned}$$

Example 7.9

Prove the identity $\frac{\sec \theta - \tan \theta}{\sec \theta + \tan \theta} = 1 - 2 \sec \theta \tan \theta + 2 \tan^2 \theta$.

Solution

$$\begin{aligned}\text{We consider } \frac{\sec \theta - \tan \theta}{\sec \theta + \tan \theta} &= \left(\frac{\sec \theta - \tan \theta}{\sec \theta + \tan \theta} \right) \times \left(\frac{\sec \theta - \tan \theta}{\sec \theta - \tan \theta} \right) \\ &= \frac{(\sec \theta - \tan \theta)^2}{\sec^2 \theta - \tan^2 \theta} \\ &= \frac{(\sec \theta - \tan \theta)^2}{1} \quad (\sec^2 \theta - \tan^2 \theta = 1) \\ &= (\sec \theta - \tan \theta)^2 = \sec^2 \theta + \tan^2 \theta - 2 \sec \theta \tan \theta \\ &= (1 + \tan^2 \theta) + \tan^2 \theta - 2 \sec \theta \tan \theta \quad (\sec^2 \theta = 1 + \tan^2 \theta) \\ &= 1 - 2 \sec \theta \tan \theta + 2 \tan^2 \theta.\end{aligned}$$

Example 7.10

Prove that $\frac{1 + \sec \theta}{\sec \theta} = \frac{\sin^2 \theta}{1 - \cos \theta}$.

Solution

$$\begin{aligned} \text{First, we consider } \frac{1 + \sec \theta}{\sec \theta} &= \frac{1 + \frac{1}{\cos \theta}}{\frac{1}{\cos \theta}} = \frac{(\cos \theta + 1)}{\cos \theta} (\cos \theta) \\ &= 1 + \cos \theta \\ &= (1 + \cos \theta) \times \frac{(1 - \cos \theta)}{(1 - \cos \theta)} \\ &= \frac{1 - \cos^2 \theta}{1 - \cos \theta} \\ &= \frac{\sin^2 \theta}{1 - \cos \theta}. \end{aligned}$$

Example 7.11

Prove the identity $(\operatorname{cosec} \theta - \sin \theta)(\sec \theta - \cos \theta) = \frac{1}{\tan \theta + \cot \theta}$.

Solution

$$\begin{aligned} \text{Now, } (\operatorname{cosec} \theta - \sin \theta)(\sec \theta - \cos \theta) &= \left(\frac{1}{\sin \theta} - \sin \theta\right)\left(\frac{1}{\cos \theta} - \cos \theta\right) \\ &= \left(\frac{1 - \sin^2 \theta}{\sin \theta}\right)\left(\frac{1 - \cos^2 \theta}{\cos \theta}\right) \\ &= \frac{\cos^2 \theta \sin^2 \theta}{\sin \theta \cos \theta} = \sin \theta \cos \theta \quad (1) \end{aligned}$$

$$\begin{aligned} \text{Next, consider } \frac{1}{\tan \theta + \cot \theta} &= \frac{1}{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}} \\ &= \frac{1}{\left(\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}\right)} \\ &= \sin \theta \cos \theta \quad (2) \end{aligned}$$

From (1) and (2), we get

$$(\operatorname{cosec} \theta - \sin \theta)(\sec \theta - \cos \theta) = \frac{1}{\tan \theta + \cot \theta}.$$

Note

$$\begin{aligned} \sin \theta \cos \theta &= \frac{\sin \theta \cos \theta}{1} \\ &= \frac{\sin \theta \cos \theta}{\sin^2 \theta + \cos^2 \theta} \\ &= \frac{1}{\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}} \\ &= \frac{1}{\frac{\sin^2 \theta}{\sin \theta \cos \theta} + \frac{\cos^2 \theta}{\sin \theta \cos \theta}} \\ &= \frac{1}{\tan \theta + \cot \theta} \end{aligned}$$

Example 7.12

If $\tan \theta + \sin \theta = m$, $\tan \theta - \sin \theta = n$ and $m \neq n$, then show that $m^2 - n^2 = 4\sqrt{mn}$.

Solution

Given that $m = \tan \theta + \sin \theta$ and $n = \tan \theta - \sin \theta$.

$$\begin{aligned}\text{Now, } m^2 - n^2 &= (\tan \theta + \sin \theta)^2 - (\tan \theta - \sin \theta)^2 \\ &= \tan^2 \theta + \sin^2 \theta + 2 \sin \theta \tan \theta - (\tan^2 \theta + \sin^2 \theta - 2 \sin \theta \tan \theta) \\ &= 4 \sin \theta \tan \theta\end{aligned}\tag{1}$$

$$\begin{aligned}\text{Also, } 4\sqrt{mn} &= 4\sqrt{(\tan \theta + \sin \theta)(\tan \theta - \sin \theta)} \\ &= 4\sqrt{\tan^2 \theta - \sin^2 \theta} = 4\sqrt{\left(\frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta\right)} \\ &= 4\sqrt{\sin^2 \theta \left(\frac{1}{\cos^2 \theta} - 1\right)} \\ &= 4\sqrt{\sin^2 \theta (\sec^2 \theta - 1)} = 4\sqrt{\sin^2 \theta \tan^2 \theta} \quad (\because \sec^2 \theta - 1 = \tan^2 \theta) \\ &= 4 \sin \theta \tan \theta.\end{aligned}\tag{2}$$

From (1) and (2), we get $m^2 - n^2 = 4\sqrt{mn}$.

Example 7.13

If $\tan^2 \alpha = \cos^2 \beta - \sin^2 \beta$, then prove that $\cos^2 \alpha - \sin^2 \alpha = \tan^2 \beta$.

Solution

Given that $\cos^2 \beta - \sin^2 \beta = \tan^2 \alpha$

$$\frac{\cos^2 \beta - \sin^2 \beta}{1} = \frac{\sin^2 \alpha}{\cos^2 \alpha}$$

$$\frac{\cos^2 \beta - \sin^2 \beta}{\cos^2 \beta + \sin^2 \beta} = \frac{\sin^2 \alpha}{\cos^2 \alpha}$$

Componendo and dividendo rule

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Applying componendo and dividendo rule, we get

$$\frac{(\cos^2 \beta - \sin^2 \beta) + (\cos^2 \beta + \sin^2 \beta)}{(\cos^2 \beta - \sin^2 \beta) - (\cos^2 \beta + \sin^2 \beta)} = \frac{\sin^2 \alpha + \cos^2 \alpha}{\sin^2 \alpha - \cos^2 \alpha}$$

$$\Rightarrow \frac{2 \cos^2 \beta}{-2 \sin^2 \beta} = \frac{1}{\sin^2 \alpha - \cos^2 \alpha}$$

$$\Rightarrow -\frac{\sin^2 \beta}{\cos^2 \beta} = \sin^2 \alpha - \cos^2 \alpha$$

$$\Rightarrow \tan^2 \beta = \cos^2 \alpha - \sin^2 \alpha, \text{ which completes the proof.}$$

Note: This problem can also be solved without using componendo and dividendo rule.

Exercise 7.1

1. Determine whether each of the following is an identity or not.
 - (i) $\cos^2 \theta + \sec^2 \theta = 2 + \sin \theta$
 - (ii) $\cot^2 \theta + \cos \theta = \sin^2 \theta$

2. Prove the following identities
 - (i) $\sec^2 \theta + \operatorname{cosec}^2 \theta = \sec^2 \theta \operatorname{cosec}^2 \theta$
 - (ii) $\frac{\sin \theta}{1 - \cos \theta} = \operatorname{cosec} \theta + \cot \theta$
 - (iii) $\sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}} = \sec \theta - \tan \theta$
 - (iv) $\frac{\cos \theta}{\sec \theta - \tan \theta} = 1 + \sin \theta$
 - (v) $\sqrt{\sec^2 \theta + \operatorname{cosec}^2 \theta} = \tan \theta + \cot \theta$
 - (vi) $\frac{1 + \cos \theta - \sin^2 \theta}{\sin \theta (1 + \cos \theta)} = \cot \theta$
 - (vii) $\sec \theta (1 - \sin \theta)(\sec \theta + \tan \theta) = 1$
 - (viii) $\frac{\sin \theta}{\operatorname{cosec} \theta + \cot \theta} = 1 - \cos \theta$

3. Prove the following identities.
 - (i) $\frac{\sin(90^\circ - \theta)}{1 + \sin \theta} + \frac{\cos \theta}{1 - \cos(90^\circ - \theta)} = 2 \sec \theta$
 - (ii) $\frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta} = 1 + \sec \theta \operatorname{cosec} \theta$
 - (iii) $\frac{\sin(90^\circ - \theta)}{1 - \tan \theta} + \frac{\cos(90^\circ - \theta)}{1 - \cot \theta} = \cos \theta + \sin \theta$
 - (iv) $\frac{\tan(90^\circ - \theta)}{\operatorname{cosec} \theta + 1} + \frac{\operatorname{cosec} \theta + 1}{\cot \theta} = 2 \sec \theta.$
 - (v) $\frac{\cot \theta + \operatorname{cosec} \theta - 1}{\cot \theta - \operatorname{cosec} \theta + 1} = \operatorname{cosec} \theta + \cot \theta.$
 - (vi) $(1 + \cot \theta - \operatorname{cosec} \theta)(1 + \tan \theta + \sec \theta) = 2$
 - (vii) $\frac{\sin \theta - \cos \theta + 1}{\sin \theta + \cos \theta - 1} = \frac{1}{\sec \theta - \tan \theta}$
 - (viii) $\frac{\tan \theta}{1 - \tan^2 \theta} = \frac{\sin \theta \sin(90^\circ - \theta)}{2 \sin^2(90^\circ - \theta) - 1}$
 - (ix) $\frac{1}{\operatorname{cosec} \theta - \cot \theta} - \frac{1}{\sin \theta} = \frac{1}{\sin \theta} - \frac{1}{\operatorname{cosec} \theta + \cot \theta}.$
 - (x) $\frac{\cot^2 \theta + \sec^2 \theta}{\tan^2 \theta + \operatorname{cosec}^2 \theta} = (\sin \theta \cos \theta)(\tan \theta + \cot \theta).$

4. If $x = a \sec \theta + b \tan \theta$ and $y = a \tan \theta + b \sec \theta$, then prove that $x^2 - y^2 = a^2 - b^2$.
5. If $\tan \theta = n \tan \alpha$ and $\sin \theta = m \sin \alpha$, then prove that $\cos^2 \theta = \frac{m^2 - 1}{n^2 - 1}$.
6. If $\sin \theta$, $\cos \theta$ and $\tan \theta$ are in G.P., then prove that $\cot^6 \theta - \cot^2 \theta = 1$.

7.3 Heights and Distances

One wonders, how the distance between planets, height of Mount Everest, distance between two objects which are far off like Earth and Sun ... , are measured or calculated. Can these be done with measuring tapes?

Of course, it is impossible to do so. Quite interestingly such distances are calculated using the idea of trigonometric ratios. These ratios are also used to construct maps, determine the position of an Island in relation to longitude and latitude.

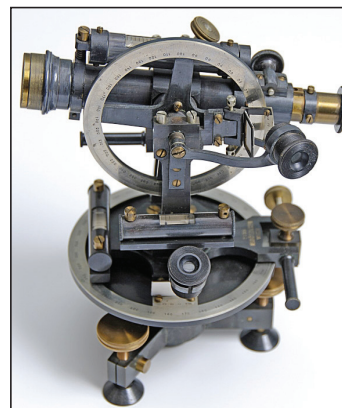


Fig. 7.2

A theodolite (Fig. 7.2) is an instrument which is used in measuring the angle between an object and the eye of the observer. A theodolite consists of two graduated wheels placed at right angles to each other and a telescope. The wheels are used for the measurement of horizontal and vertical angles. The angle to the desired point is measured by positioning the telescope towards that point. The angle can be read on the telescopic scale.

Suppose we wish to find the height of our school flag post without actually measuring it.

Assume that a student stands on the ground at point A , which is 10 m away from the foot B of the flag post. He observes the top of the flag post at an angle of 60° . Suppose that the height of his eye level E from the ground level is 1.2 m. (see fig no.7.3)

In the right angled $\triangle DEC$, $\angle DEC = 60^\circ$.

$$\text{Now,} \quad \tan 60^\circ = \frac{CD}{EC}$$

$$\Rightarrow \quad CD = EC \tan 60^\circ$$

$$\begin{aligned} \text{Thus,} \quad CD &= 10\sqrt{3} = 10 \times 1.732 \\ &= 17.32 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Hence, the height of the flag post, } BD &= BC + CD \\ &= 1.2 + 17.32 = 18.52 \text{ m} \end{aligned}$$

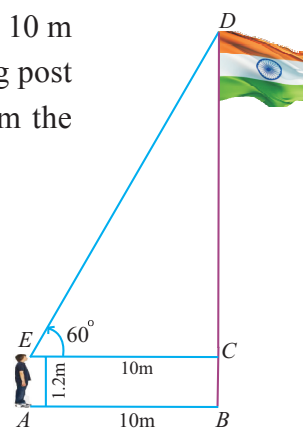


Fig. 7.3

Thus, we are able to find the height of our school flag post without actually measuring it. So, in a right triangle, if one side and one acute angle are known, we can find the other sides of the triangle using trigonometrical ratios. Let us define a few terms which we use very often in finding the heights and distances.

Line of sight

If we are viewing an object, the **line of sight** is a straight line from our eye to the object. Here we treat the object as a point since distance involved is quite large.

Angle of depression and angle of elevation

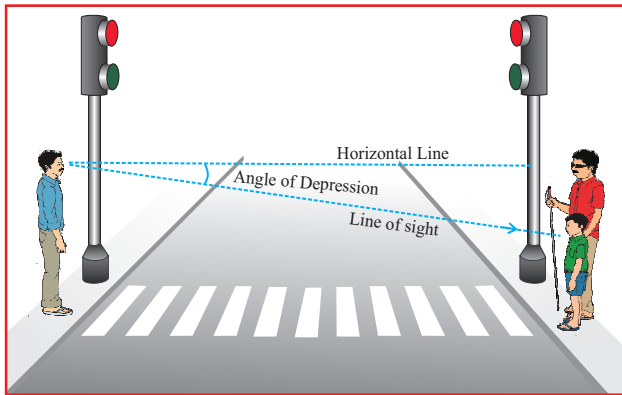


Fig. 7.4

If an object is below the horizontal line from the eye, we have to lower our head to view the object. In this process our eyes move through an angle. This angle is called the **angle of depression**, That is, the angle of depression of an object viewed is the angle formed by the line of sight with the horizontal line, when the object is below the horizontal line (See Fig. 7.4).

If an object is above the horizontal line from our eyes we have to raise our head to view the object. In this process our eyes move through an angle formed by the line of sight and horizontal line which is called the **angle of elevation**. (See Fig. 7.5).

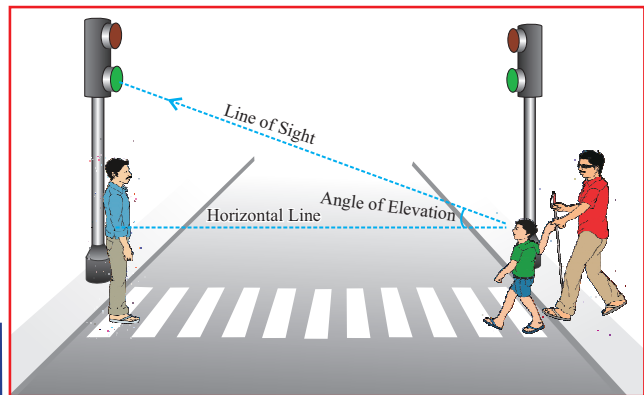


Fig. 7.5

Note

- (i) An observer is taken as a point if the height of the observer is not given.
- (ii) The angle of elevation of an object as seen by the observer is same as the angle of depression of the observer as seen from the object.

To solve problems involving heights and distances, the following strategy may be useful

- (i) Read the statements of the question carefully and draw a rough diagram accordingly.
- (ii) Label the diagram and mark the given values.
- (iii) Denote the unknown dimension, say ' h ' when the height is to be calculated and ' x ' when the distance is to be calculated.
- (iv) Identify the trigonometrical ratio that will be useful for solving the problem.
- (v) Substitute the given values and solve for unknown.

The following activity may help us learn how to measure the height of an object which will be difficult to measure otherwise.

Activity

- Tie one end of a string to the middle of a straw and the other end of the string to a paper clip.
- Glue this straw to the base of a protractor so that the middle of the straw aligns with the centre of the protractor. Make sure that the string hangs freely to create a vertical line or the plumb-line.
- Find an object outside that is too tall to measure directly, such as a basket ball hoop, a flagpole, or the school building.
- Look at the top of the object through the straw. Find the angle where the string and protractor intersect. Determine the angle of elevation by subtracting this measurement from 90° . Let it be θ .
- Measure the distance from your eye level to the ground and from your foot to the base of the object that you are measuring, say y .
- Make a sketch of your measurements.
- To find the height (h) of the object, use the following equation.

$$h = x + y \tan \theta$$
, where x represents the distance from your eye level to the ground.

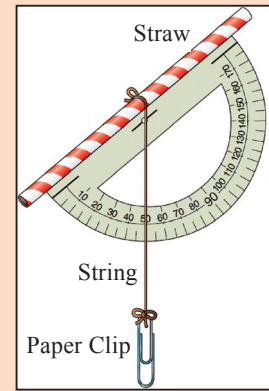


Fig. 7.6

Example 7.14

A kite is flying with a string of length 200 m. If the thread makes an angle 30° with the ground, find the distance of the kite from the ground level. (Here, assume that the string is along a straight line)

Solution Let h denote the distance of the kite from the ground level.

In the figure, AC is the string

Given that $\angle CAB = 30^\circ$ and $AC = 200$ m

In the right $\triangle CAB$, $\sin 30^\circ = \frac{h}{200}$

$$\Rightarrow h = 200 \sin 30^\circ$$

$$\therefore h = 200 \times \frac{1}{2} = 100 \text{ m}$$

Hence, the distance of the kite from the ground level is 100 m.

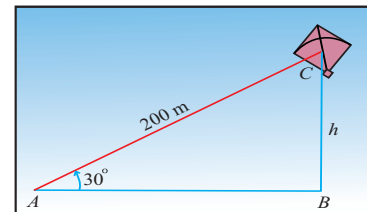


Fig. 7.7

Example 7.15

A ladder leaning against a vertical wall, makes an angle of 60° with the ground. The foot of the ladder is 3.5 m away from the wall. Find the length of the ladder.

Solution Let AC denote the ladder and B be the foot of the wall.

Let the length of the ladder AC be x metres.

Given that $\angle CAB = 60^\circ$ and $AB = 3.5$ m.

$$\begin{aligned} \text{In the right } \triangle CAB, \quad \cos 60^\circ &= \frac{AB}{AC} \\ \implies AC &= \frac{AB}{\cos 60^\circ} \\ \therefore x &= 2 \times 3.5 = 7 \text{ m} \end{aligned}$$

Thus, the length of the ladder is 7 m.

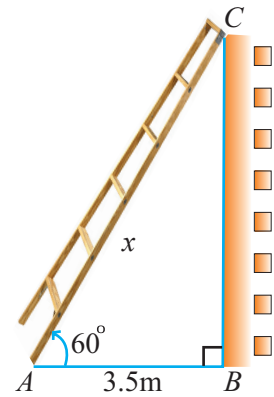


Fig. 7.8

Example 7.16

Find the angular elevation (angle of elevation from the ground level) of the Sun when the length of the shadow of a 30 m long pole is $10\sqrt{3}$ m.

Solution Let S be the position of the Sun and BC be the pole.

Let AB denote the length of the shadow of the pole.

Let the angular elevation of the Sun be θ .

Given that $AB = 10\sqrt{3}$ m and

$$BC = 30 \text{ m}$$

$$\begin{aligned} \text{In the right } \triangle CAB, \quad \tan \theta &= \frac{BC}{AB} = \frac{30}{10\sqrt{3}} = \frac{3}{\sqrt{3}} \\ \implies \tan \theta &= \sqrt{3} \\ \therefore \theta &= 60^\circ \end{aligned}$$

Thus, the angular elevation of the Sun from the ground level is 60° .

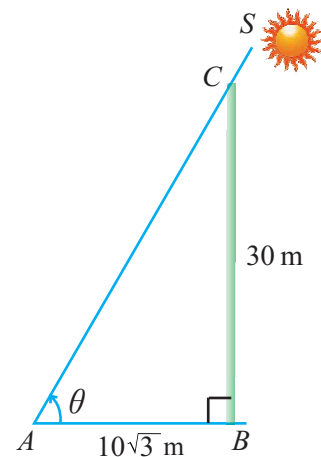


Fig. 7.9

Example 7.17

The angle of elevation of the top of a tower as seen by an observer is 30° . The observer is at a distance of $30\sqrt{3}$ m from the tower. If the eye level of the observer is 1.5 m above the ground level, then find the height of the tower.

Solution Let BD be the height of the tower and AE be the distance of the eye level of the observer from the ground level.

Draw EC parallel to AB such that $AB = EC$.

Given $AB = EC = 30\sqrt{3}$ m and

$$AE = BC = 1.5 \text{ m}$$

In right angled $\triangle DEC$,

$$\tan 30^\circ = \frac{CD}{EC}$$

$$\Rightarrow CD = EC \tan 30^\circ = \frac{30\sqrt{3}}{\sqrt{3}}$$

$$\therefore CD = 30 \text{ m}$$

Thus, the height of the tower,

$$BD = BC + CD$$

$$= 1.5 + 30 = 31.5 \text{ m.}$$

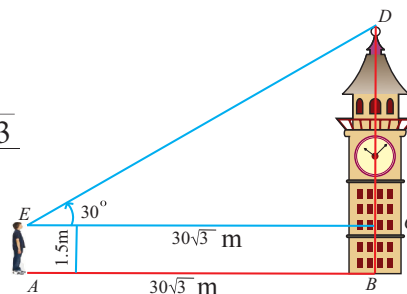


Fig. 7.10

Example 7.18

A vertical tree is broken by the wind. The top of the tree touches the ground and makes an angle 30° with it. If the top of the tree touches the ground 30 m away from its foot, then find the actual height of the tree.

Solution Let C be the point at which the tree is broken and let the top of the tree touch the ground at A .

Let B denote the foot of the tree.

Given $AB = 30 \text{ m}$ and

$$\angle CAB = 30^\circ.$$

In the right angled $\triangle CAB$,

$$\tan 30^\circ = \frac{BC}{AB}$$

$$\Rightarrow BC = AB \tan 30^\circ$$

$$\therefore BC = \frac{30}{\sqrt{3}}$$

$$= 10\sqrt{3} \text{ m}$$

(1)

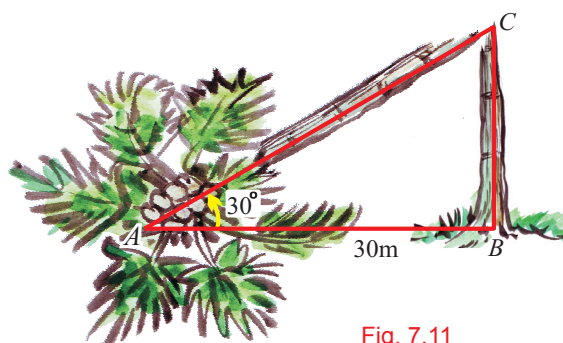


Fig. 7.11

Now,

$$\cos 30^\circ = \frac{AB}{AC}$$

$$\Rightarrow AC = \frac{AB}{\cos 30^\circ}$$

So,

$$AC = \frac{30 \times 2}{\sqrt{3}} = 10\sqrt{3} \times 2 = 20\sqrt{3} \text{ m.}$$

(2)

Thus, the height of the tree

$$= BC + AC = 10\sqrt{3} + 20\sqrt{3}$$

$$= 30\sqrt{3} \text{ m.}$$

Example 7.19

A jet fighter at a height of 3000 m from the ground, passes directly over another jet fighter at an instance when their angles of elevation from the same observation point are 60° and 45° respectively. Find the distance of the first jet fighter from the second jet at that instant.

($\sqrt{3} = 1.732$)

Solution Let O be the point of observation.

Let A and B be the positions of the two jet fighters at the given instant when one is directly above the other.

Let C be the point on the ground such that $AC = 3000$ m.

Given $\angle AOC = 60^\circ$ and $\angle BOC = 45^\circ$

Let h denote the distance between the jets at the instant.

In the right angled $\triangle BOC$, $\tan 45^\circ = \frac{BC}{OC}$

$$\Rightarrow OC = BC \quad (\because \tan 45^\circ = 1)$$

Thus, $OC = 3000 - h$ (1)

In the right angled $\triangle AOC$, $\tan 60^\circ = \frac{AC}{OC}$

$$\begin{aligned} \Rightarrow OC &= \frac{AC}{\tan 60^\circ} = \frac{3000}{\sqrt{3}} \\ &= \frac{3000}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 1000\sqrt{3} \end{aligned} \quad (2)$$

From (1) and (2), we get $3000 - h = 1000\sqrt{3}$

$$\Rightarrow h = 3000 - 1000 \times 1.732 = 1268 \text{ m}$$

The distance of the first jet fighter from the second jet at that instant is 1268 m.

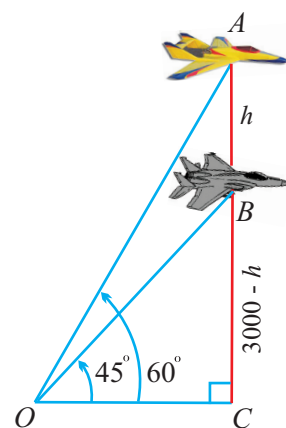


Fig. 7.12

Example 7.20

The angle of elevation of the top of a hill from the foot of a tower is 60° and the angle of elevation of the top of the tower from the foot of the hill is 30° .

If the tower is 50 m high, then find the height of the hill.

Solution Let AD be the height of tower and BC be the height of the hill.

Given $\angle CAB = 60^\circ$, $\angle ABD = 30^\circ$ and $AD = 50$ m.

Let $BC = h$ metres.

Now, in the right angled $\triangle DAB$, $\tan 30^\circ = \frac{AD}{AB}$

$$\Rightarrow AB = \frac{AD}{\tan 30^\circ}$$

$$\therefore AB = 50\sqrt{3} \text{ m} \quad (1)$$

Also, in the right angled $\triangle CAB$, $\tan 60^\circ = \frac{BC}{AB}$

$$\Rightarrow BC = AB \tan 60^\circ$$

Thus, using (1) we get $h = BC = (50\sqrt{3})\sqrt{3} = 150$ m

Hence, the height of the hill is 150 m.

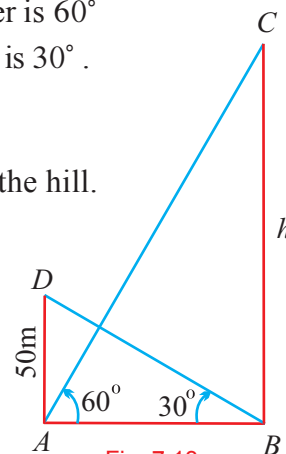


Fig. 7.13

Example 7.21

A vertical wall and a tower are on the ground. As seen from the top of the tower, the angles of depression of the top and bottom of the wall are 45° and 60° respectively. Find the height of the wall if the height of the tower is 90 m. ($\sqrt{3} = 1.732$)

Solution Let AE denote the wall and BD denote the tower.

Draw EC parallel to AB such that $AB=EC$. Thus, $AE = BC$.

Let $AB = x$ metres and $AE = h$ metres.

Given that $BD = 90$ m and $\angle DAB = 60^\circ$, $\angle DEC = 45^\circ$.

Now, $AE = BC = h$ metres

Thus, $CD = BD - BC = 90 - h$.

In the right angled $\triangle DAB$, $\tan 60^\circ = \frac{BD}{AB} = \frac{90}{x}$

$$\Rightarrow x = \frac{90}{\sqrt{3}} = 30\sqrt{3} \quad (1)$$

In the right angled $\triangle DEC$, $\tan 45^\circ = \frac{DC}{EC} = \frac{90 - h}{x}$

$$\text{Thus, } x = 90 - h \quad (2)$$

From (1) and (2), we have $90 - h = 30\sqrt{3}$

Thus, the height of the wall, $h = 90 - 30\sqrt{3} = 38.04$ m

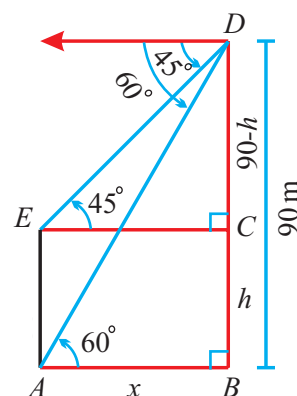


Fig. 7.14

Example 7.22

A girl standing on a lighthouse built on a cliff near the seashore, observes two boats due East of the lighthouse. The angles of depression of the two boats are 30° and 60° . The distance between the boats is 300 m. Find the distance of the top of the lighthouse from the sea level.

Solution Let A and D denote the foot of the cliff and the top of the lighthouse respectively.

Let B and C denote the two boats.

Let h metres be the distance of the top of the lighthouse from the sea level.

Let $AB = x$ metres.

Given that $\angle ABD = 60^\circ$, $\angle ACD = 30^\circ$

In the right angled $\triangle ABD$,

$$\tan 60^\circ = \frac{AD}{AB}$$

$$\Rightarrow AB = \frac{AD}{\tan 60^\circ}$$

$$\text{Thus, } x = \frac{h}{\sqrt{3}} \quad (1)$$

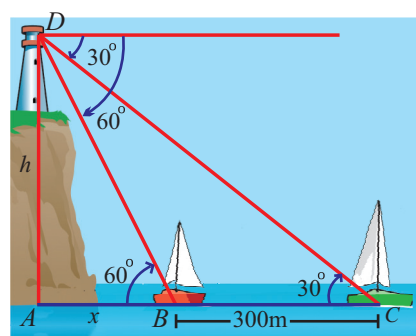


Fig. 7.15

Also, in the right angled $\triangle ACD$, we have

$$\begin{aligned} \tan 30^\circ &= \frac{AD}{AC} \\ \Rightarrow AC &= \frac{AD}{\tan 30^\circ} \Rightarrow x + 300 = \frac{h}{\left(\frac{1}{\sqrt{3}}\right)} \end{aligned}$$

Thus, $x + 300 = h\sqrt{3}$. (2)

Using (1) in (2), we get $\frac{h}{\sqrt{3}} + 300 = h\sqrt{3}$

$$\begin{aligned} \Rightarrow h\sqrt{3} - \frac{h}{\sqrt{3}} &= 300 \\ \therefore 2h &= 300\sqrt{3}. \quad \text{Thus, } h = 150\sqrt{3}. \end{aligned}$$

Hence, the height of the lighthouse from the sea level is $150\sqrt{3}$ m.

Example 7.23

A boy spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground level. The distance of his eye level from the ground is 1.2 m. The angle of elevation of the balloon from his eyes at an instant is 60° . After some time, from the same point of observation, the angle of elevation of the balloon reduces to 30° . Find the distance covered by the balloon during the interval.

Solution Let A be the point of observation.

Let E and D be the positions of the balloon when its angles of elevation are 60° and 30° respectively.

Let B and C be the points on the horizontal line such that $BE = CD$.

Let A' , B' and C' be the points on the ground such that

$$A'A = B'B = C'C = 1.2 \text{ m.}$$

Given that $\angle EAB = 60^\circ$, $\angle DAC = 30^\circ$

$$BB' = CC' = 1.2 \text{ m and } C'D = 88.2 \text{ m.}$$

Also, we have $BE = CD = 87$ m.

Now, in the right angled $\triangle EAB$, we have

$$\tan 60^\circ = \frac{BE}{AB}$$

Thus, $AB = \frac{87}{\tan 60^\circ} = \frac{87}{\sqrt{3}} = 29\sqrt{3}$

Again in the right angled $\triangle DAC$, we have $\tan 30^\circ = \frac{DC}{AC}$

Thus, $AC = \frac{87}{\tan 30^\circ} = 87\sqrt{3}$.

Therefore, the distance covered by the balloon is

$$\begin{aligned} ED = BC &= AC - AB \\ &= 87\sqrt{3} - 29\sqrt{3} = 58\sqrt{3} \text{ m.} \end{aligned}$$

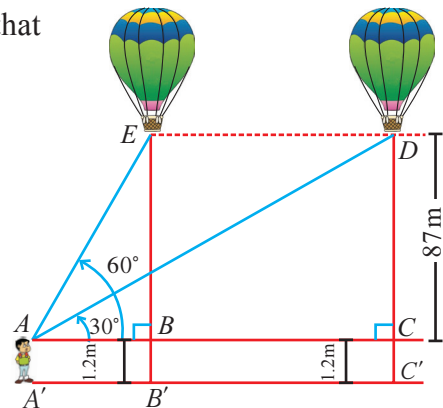


Fig. 7.16

Example 7.24

A flag post stands on the top of a building. From a point on the ground, the angles of elevation of the top and bottom of the flag post are 60° and 45° respectively. If the height of the flag post is 10 m, find the height of the building. ($\sqrt{3} = 1.732$)

Solution

Let A be the point of observation and B be the foot of the building.

Let BC denote the height of the building and CD denote height of the flag post.

Given that $\angle CAB = 45^\circ$, $\angle DAB = 60^\circ$ and $CD = 10$ m

Let $BC = h$ metres and $AB = x$ metres.

Now, in the right angled $\triangle CAB$,

$$\tan 45^\circ = \frac{BC}{AB}.$$

$$\text{Thus, } AB = BC \quad \text{i.e., } x = h \quad (1)$$

Also, in the right angled $\triangle DAB$,

$$\tan 60^\circ = \frac{BD}{AB}$$

$$\Rightarrow AB = \frac{h + 10}{\tan 60^\circ} \Rightarrow x = \frac{h + 10}{\sqrt{3}} \quad (2)$$

$$\text{From (1) and (2), we get } h = \frac{h + 10}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3}h - h = 10$$

$$\begin{aligned} \Rightarrow h &= \left(\frac{10}{\sqrt{3} - 1} \right) \left(\frac{\sqrt{3} + 1}{\sqrt{3} + 1} \right) = \frac{10(\sqrt{3} + 1)}{3 - 1} \\ &= 5(2.732) = 13.66 \text{ m} \end{aligned}$$

Hence, the height of the building is 13.66 m.

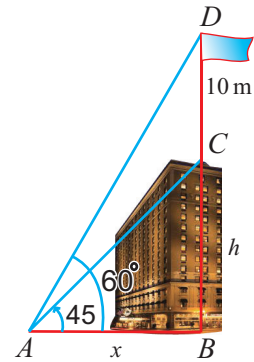


Fig. 7.17

Example 7.25

A man on the deck of a ship, 14 m above the water level, observes that the angle of elevation of the top of a cliff is 60° and the angle of depression of the base of the cliff is 30° . Find the height of the cliff.

Solution Let BD be the height of the cliff.

Let A be the position of ship and E be the point of observation so that $AE = 14$ m.

Draw EC parallel to AB such that $AB = EC$.

Given that $\angle ABE = 30^\circ$, $\angle DEC = 60^\circ$

In the right angled $\triangle ABE$, $\tan 30^\circ = \frac{AE}{AB}$

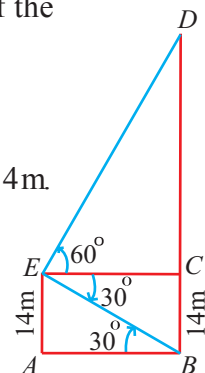


Fig. 7.18

$$\therefore AB = \frac{AE}{\tan 30^\circ} \implies AB = 14\sqrt{3}$$

Thus, $EC = 14\sqrt{3}$ ($\because AB = EC$)

In the right angled $\triangle DEC$, $\tan 60^\circ = \frac{CD}{EC}$

$$\therefore CD = EC \tan 60^\circ \implies CD = (14\sqrt{3})\sqrt{3} = 42 \text{ m}$$

Thus, the height of the cliff, $BD = BC + CD = 14 + 42 = 56 \text{ m}$.

Example 7.26

The angle of elevation of an aeroplane from a point A on the ground is 60° . After a flight of 15 seconds horizontally, the angle of elevation changes to 30° . If the aeroplane is flying at a speed of 200 m/s, then find the constant height at which the aeroplane is flying.

Solution Let A be the point of observation.

Let E and D be positions of the aeroplane initially and after 15 seconds respectively.

Let BE and CD denote the constant height at which the aeroplane is flying.

Given that $\angle DAC = 30^\circ$, $\angle EAB = 60^\circ$.

Let $BE = CD = h$ metres.

Let $AB = x$ metres.

The distance covered in 15 seconds,

$$ED = 200 \times 15 = 3000 \text{ m} \quad (\text{distance travelled} = \text{speed} \times \text{time})$$

Thus, $BC = 3000 \text{ m}$.

In the right angled $\triangle DAC$,

$$\tan 30^\circ = \frac{CD}{AC}$$

$$\implies CD = AC \tan 30^\circ$$

$$\text{Thus, } h = (x + 3000) \frac{1}{\sqrt{3}}. \quad (1)$$

In the right angled $\triangle EAB$,

$$\tan 60^\circ = \frac{BE}{AB}$$

$$\implies BE = AB \tan 60^\circ \implies h = \sqrt{3} x \quad (2)$$

$$\text{From (1) and (2), we have } \sqrt{3} x = \frac{1}{\sqrt{3}}(x + 3000)$$

$$\implies 3x = x + 3000 \implies x = 1500 \text{ m.}$$

Thus, from (2) it follows that $h = 1500\sqrt{3} \text{ m}$.

The constant height at which the aeroplane is flying, is $1500\sqrt{3} \text{ m}$.

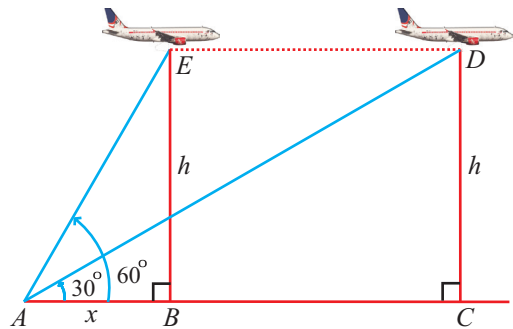
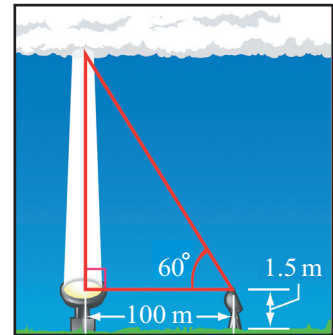


Fig. 7.19

Exercise 7.2

1. A ramp for unloading a moving truck, has an angle of elevation of 30° . If the top of the ramp is 0.9 m above the ground level, then find the length of the ramp.
2. A girl of height 150 cm stands in front of a lamp-post and casts a shadow of length $150\sqrt{3}$ cm on the ground. Find the angle of elevation of the top of the lamp-post .
3. Suppose two insects A and B can hear each other up to a range of 2 m. The insect A is on the ground 1 m away from a wall and sees her friend B on the wall, about to be eaten by a spider. If A sounds a warning to B and if the angle of elevation of B from A is 30° , will the spider have a meal or not ? (Assume that B escapes if she hears A calling)

4. To find the cloud ceiling, one night an observer directed a spotlight vertically at the clouds. Using a theodolite placed 100 m from the spotlight and 1.5 m above the ground, he found the angle of elevation to be 60° . How high was the cloud ceiling? (Hint : See figure)



(**Note:** Cloud ceiling is the lowest altitude at which solid cloud is present. The cloud ceiling at airports must be sufficiently high for safe take offs and landings. At night the cloud ceiling can be determined by illuminating the base of the clouds by a spotlight pointing vertically upward.)

5. A simple pendulum of length 40 cm subtends 60° at the vertex in one full oscillation. What will be the shortest distance between the initial position and the final position of the bob? (between the extreme ends)
6. Two crows A and B are sitting at a height of 15 m and 10 m in two different trees vertically opposite to each other . They view a vadai (an eatable) on the ground at an angle of depression 45° and 60° respectively. They start at the same time and fly at the same speed along the shortest path to pick up the vadai. Which bird will succeed in it?
7. A lamp-post stands at the centre of a circular park. Let P and Q be two points on the boundary such that PQ subtends an angle 90° at the foot of the lamp-post and the angle of elevation of the top of the lamp post from P is 30° . If $PQ = 30$ m, then find the height of the lamp post.
8. A person in an helicopter flying at a height of 700 m, observes two objects lying opposite to each other on either bank of a river. The angles of depression of the objects are 30° and 45° . Find the width of the river. ($\sqrt{3} = 1.732$)
9. A person X standing on a horizontal plane, observes a bird flying at a distance of 100 m from him at an angle of elevation of 30° . Another person Y standing on the roof of a 20 m high building, observes the bird at the same time at an angle of elevation of 45° . If X and Y are on the opposite sides of the bird, then find the distance of the bird from Y .

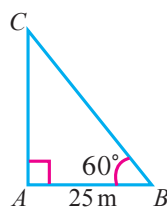
10. A student sitting in a classroom sees a picture on the black board at a height of 1.5 m from the horizontal level of sight. The angle of elevation of the picture is 30° . As the picture is not clear to him, he moves straight towards the black board and sees the picture at an angle of elevation of 45° . Find the distance moved by the student.
11. A boy is standing at some distance from a 30 m tall building and his eye level from the ground is 1.5 m. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.
12. From the top of a lighthouse of height 200 feet, the lighthouse keeper observes a Yacht and a Barge along the same line of sight. The angles of depression for the Yacht and the Barge are 45° and 30° respectively. For safety purposes the two sea vessels should be atleast 300 feet apart. If they are less than 300 feet, the keeper has to sound the alarm. Does the keeper have to sound the alarm?
13. A boy standing on the ground, spots a balloon moving with the wind in a horizontal line at a constant height. The angle of elevation of the balloon from the boy at an instant is 60° . After 2 minutes, from the same point of observation, the angle of elevation reduces to 30° . If the speed of wind is $29\sqrt{3}$ m/min. then, find the height of the balloon from the ground level.
14. A straight highway leads to the foot of a tower. A man standing on the top of the tower spots a van at an angle of depression of 30° . The van is approaching the tower with a uniform speed. After 6 minutes, the angle of depression of the van is found to be 60° . How many more minutes will it take for the van to reach the tower?
15. The angles of elevation of an artificial earth satellite is measured from two earth stations, situated on the same side of the satellite, are found to be 30° and 60° . The two earth stations and the satellite are in the same vertical plane. If the distance between the earth stations is 4000 km, find the distance between the satellite and earth. ($\sqrt{3} = 1.732$)
16. From the top of a tower of height 60 m, the angles of depression of the top and the bottom of a building are observed to be 30° and 60° respectively. Find the height of the building.
17. From the top and foot of a 40 m high tower, the angles of elevation of the top of a lighthouse are found to be 30° and 60° respectively. Find the height of the lighthouse. Also find the distance of the top of the lighthouse from the foot of the tower.
18. The angle of elevation of a hovering helicopter as seen from a point 45 m above a lake is 30° and the angle of depression of its reflection in the lake, as seen from the same point and at the same time, is 60° . Find the distance of the helicopter from the surface of the lake.

Exercise 7.3

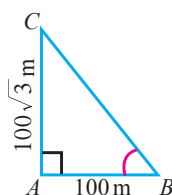
Choose the correct answer

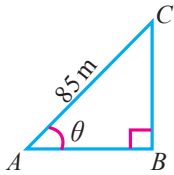
1. $(1 - \sin^2 \theta) \sec^2 \theta =$
 (A) 0 (B) 1 (C) $\tan^2 \theta$ (D) $\cos^2 \theta$
2. $(1 + \tan^2 \theta) \sin^2 \theta =$
 (A) $\sin^2 \theta$ (B) $\cos^2 \theta$ (C) $\tan^2 \theta$ (D) $\cot^2 \theta$
3. $(1 - \cos^2 \theta)(1 + \cot^2 \theta) =$
 (A) $\sin^2 \theta$ (B) 0 (C) 1 (D) $\tan^2 \theta$
4. $\sin(90^\circ - \theta) \cos \theta + \cos(90^\circ - \theta) \sin \theta =$
 (A) 1 (B) 0 (C) 2 (D) -1
5. $1 - \frac{\sin^2 \theta}{1 + \cos \theta} =$
 (A) $\cos \theta$ (B) $\tan \theta$ (C) $\cot \theta$ (D) $\operatorname{cosec} \theta$
6. $\cos^4 x - \sin^4 x =$
 (A) $2 \sin^2 x - 1$ (B) $2 \cos^2 x - 1$ (C) $1 + 2 \sin^2 x$ (D) $1 - 2 \cos^2 x$
7. If $\tan \theta = \frac{a}{x}$, then the value of $\frac{x}{\sqrt{a^2 + x^2}} =$
 (A) $\cos \theta$ (B) $\sin \theta$ (C) $\operatorname{cosec} \theta$ (D) $\sec \theta$
8. If $x = a \sec \theta$, $y = b \tan \theta$, then the value of $\frac{x^2}{a^2} - \frac{y^2}{b^2} =$
 (A) 1 (B) -1 (C) $\tan^2 \theta$ (D) $\operatorname{cosec}^2 \theta$
9. $\frac{\sec \theta}{\cot \theta + \tan \theta} =$
 (A) $\cot \theta$ (B) $\tan \theta$ (C) $\sin \theta$ (D) $-\cot \theta$
10. $\frac{\sin(90^\circ - \theta) \sin \theta}{\tan \theta} + \frac{\cos(90^\circ - \theta) \cos \theta}{\cot \theta} =$
 (A) $\tan \theta$ (B) 1 (C) -1 (D) $\sin \theta$

11. In the adjoining figure, $AC =$
 (A) 25 m (B) $25\sqrt{3}$ m
 (C) $\frac{25}{\sqrt{3}}$ m (D) $25\sqrt{2}$ m



12. In the adjoining figure $\angle ABC =$
 (A) 45° (B) 30°
 (C) 60° (D) 50°



13. A man is 28.5 m away from a tower. His eye level above the ground is 1.5 m. The angle of elevation of the tower from his eyes is 45° . Then the height of the tower is
 (A) 30 m (B) 27.5 m (C) 28.5 m (D) 27 m
14. In the adjoining figure, $\sin \theta = \frac{15}{17}$. Then $BC =$
 (A) 85 m (B) 65 m
 (C) 95 m (D) 75 m
- 
15. $(1 + \tan^2 \theta)(1 - \sin \theta)(1 + \sin \theta) =$
 (A) $\cos^2 \theta - \sin^2 \theta$ (B) $\sin^2 \theta - \cos^2 \theta$
 (C) $\sin^2 \theta + \cos^2 \theta$ (D) 0
16. $(1 + \cot^2 \theta)(1 - \cos \theta)(1 + \cos \theta) =$
 (A) $\tan^2 \theta - \sec^2 \theta$ (B) $\sin^2 \theta - \cos^2 \theta$
 (C) $\sec^2 \theta - \tan^2 \theta$ (D) $\cos^2 \theta - \sin^2 \theta$
17. $(\cos^2 \theta - 1)(\cot^2 \theta + 1) + 1 =$
 (A) 1 (B) -1 (C) 2 (D) 0
18. $\frac{1 + \tan^2 \theta}{1 + \cot^2 \theta} =$
 (A) $\cos^2 \theta$ (B) $\tan^2 \theta$ (C) $\sin^2 \theta$ (D) $\cot^2 \theta$
19. $\sin^2 \theta + \frac{1}{1 + \tan^2 \theta} =$
 (A) $\operatorname{cosec}^2 \theta + \cot^2 \theta$ (B) $\operatorname{cosec}^2 \theta - \cot^2 \theta$
 (C) $\cot^2 \theta - \operatorname{cosec}^2 \theta$ (D) $\sin^2 \theta - \cos^2 \theta$
20. $9 \tan^2 \theta - 9 \sec^2 \theta =$
 (A) 1 (B) 0 (C) 9 (D) -9

Do you know?

Paul Erdos (26th March, 1913 – 20th September, 1996) was a Hungarian Mathematician. Erdos was one of the most prolific publishers of research articles in mathematical history, comparable only with **Leonhard Euler**. He wrote around 1,475 mathematical articles in his life lifetime, while Euler credited with approximately 800 research articles. He strongly believed in and practised mathematics as a social activity, having 511 different collaborators in his lifetime.