BUSINESS MATHEMATICS

Higher Secondary - First Year

Untouchability is a sin
Untouchability is a crime
Untouchability is inhuman

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Preface

This book on Business Mathematics has been written in conformity with the revised syllabus for the first year of the Higher Secondary classes.

The aim of this text book is to provide the students with the basic knowledge in the subject. We have given in the book the Definitions, Theorems and Observations, followed by typical problems and the step by step solution. The society's increasing business orientation and the students' preparedness to meet the future needs have been taken care of in this book on Business Mathematics.

This book aims at an exhaustive coverage of the curriculum and there is definitely an attempt to kindle the students creative ability.

While preparing for the examination students should not restrict themselves only to the questions / problems given in the self evaluation. They must be prepared to answer the questions and problems from the entire text.

We welcome suggestions from students, teachers and academicians so that this book may further be improved upon.

We thank everyone who has lent a helping hand in the preparation of this book.

Chairperson
The Text Book Committee

SYLLABUS

1) Matrices and Determinants (15 periods)

Order - Types of matrices - Addition and subtraction of matrices and Multiplication of a matrix by a scalar - Product of matrices.

Evaluation of determinants of order two and three - Properties of determinants (Statements only) - Singular and non singular matrices - Product of two determinants.

2) Algebra (20 periods)

Partial fractions - Linear non repeated and repeated factors - Quadratic non repeated types. Permutations - Applications - Permutation of repeated objects - Circular permutaion. Combinations - Applications - Mathematical induction - Summation of series using Σn, Σn^2 and Σn^3. Binomial theorem for a positive integral index - Binomial coefficients.

3) Sequences and series (20 periods)

Harnomic progression - Means of two positive real numbers - Relation between A.M., G.M., and H.M. - Sequences in general - Specifying a sequence by a rule and by a recursive relation - Compound interest - Nominal rate and effective rate - Annuities - immediate and due.

4) Analytical Geometry (30 periods)

Locus - Straight lines - Normal form, symmetric form - Length of perpendicular from a point to a line - Equation of the bisectors of the angle between two lines - Perpendicular and parallel lines - Concurrent lines - Circle - Centre radius form - Diameter form - General form - Length of tangent from a point to a circle - Equation of tangent - Chord of contact of tangents.

5) Trigonometry (25 periods)

Standard trigonometric identities - Signs of trigonometric ratios - compound angles - Addition formulae - Multiple and submultiple angles - Product formulae - Principal solutions - Trigonometric equations of the form $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$ and $\tan\theta = \tan\alpha$ - Inverse trigonometric functions.

6) Functions and their Graphs (15 Periods)

Functions of a real value - Constants and variables - Neighbourhood - Representation of functions - Tabular and graphical form - Vertical line test for functions - Linear functions - Determination of slopes - Power function - 2^x and e^x - Circular functions - Graphs of sinx, ,cosx and tanx - Arithmetics of functions (sum, difference, product and quotient) Absolute value function, signum function - Step function - Inverse of a function - Even and odd functions - Composition of functions

7) Differential calculus (30 periods)

Limit of a function - Standard forms

$\underset{x \to a}{Lt} \frac{x^n - a^n}{x-a}$, $\underset{x \to 0}{Lt} (1+\frac{1}{x})^x$, $\underset{x \to 0}{Lt} \frac{e^x - 1}{x}$, $\underset{x \to 0}{Lt} \frac{\log(1+x)}{x}$,

$\underset{x \to 0}{Lt} \frac{\sin\theta}{\theta}$ (statement only)

Continuity of functions - Graphical interpretation - Differentiation - Geometrical interpretation - Differtentiation using first principles - Rules of differentiation - Chain rule - Logarithmic Differentitation - Differentiation of implicit functions - parametric functions - Second order derivatives.

8) Integral calculus (25 periods)

Integration - Methods of integration - Substitution - Standard forms - integration by parts - Definite integral - Integral as the limit of an infinite sum (statement only).

9) Stocks, Shares and Debentures (15 periods)

Basic concepts - Distinction between shares and debentures - Mathematical aspects of purchase and sale of shares - Debentures with nominal rate.

10) Statistics (15 Periods)

Measures of central tendency for a continuous frequency distribution Mean, Median, Mode Geometric Mean and Harmonic Mean - Measures of dispersion for a continuous frequency distribution - Range - Standard deviation - Coefficient of variation - Probability - Basic concepts - Axiomatic approach - Classical definition - Basic theorems - Addition theorem (statement only) - Conditional probability - Multiplication theorem (statement only) - Baye's theorem (statement only) - Simple problems.

Contents

MATRICES AND DETERMINANTS 1

1.1 MATRIX ALGEBRA

Sir ARTHUR CAYLEY (1821-1895) of England was the first Mathematician to introduce the term MATRIX in the year 1858. But in the present day applied Mathematics in overwhelmingly large majority of cases it is used, as a notation to represent a large number of simultaneous equations in a compact and convenient manner.

Matrix Theory has its applications in Operations Research, Economics and Psychology. Apart from the above, matrices are now indispensible in all branches of Engineering, Physical and Social Sciences, Business Management, Statistics and Modern Control systems.

1.1.1 Definition of a Matrix

A rectangular array of numbers or functions represented by the symbol

$$\begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{m1} & a_{m2} & \dots a_{mn} \end{pmatrix} \text{ is called a MATRIX}$$

The numbers or functions a_{ij} of this array are called elements, may be real or complex numbers, where as m and n are positive integers, which denotes the number of Rows and number of Columns.

For example

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} x^2 & \sin x \\ \sqrt{x} & \frac{1}{x} \end{pmatrix} \text{ are the matrices}$$

1.1.2 Order of a Matrix

A matrix A with m rows and n columns is said to be of the order m by n (m x n).

Symbolically

$A = (a_{ij})_{m \times n}$ is a matrix of order m x n. The first subscript i in (a_{ij}) ranging from 1 to m identifies the rows and the second subscript j in (a_{ij}) ranging from 1 to n identifies the columns.

For example

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is a Matrix of order 2 x 3 and

$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a Matrix of order 2 x 2

$C = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$ is a Matrix of order 2 x 2

$D = \begin{pmatrix} 0 & 22 & 30 \\ -4 & 5 & -67 \\ 78 & -8 & 93 \end{pmatrix}$ is a Matrix of order 3 x 3

1.1.3 Types of Matrices

(i) SQUARE MATRIX

When the number of rows is equal to the number of columns, the matrix is called a Square Matrix.

For example

$A = \begin{pmatrix} 5 & 7 \\ 6 & 3 \end{pmatrix}$ is a Square Matrix of order 2

$B = \begin{pmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \\ 2 & 4 & 9 \end{pmatrix}$ is a Square Matrix of order 3

$C = \begin{pmatrix} \sin\alpha & \sin\beta & \sin\delta \\ \cos\alpha & \cos\beta & \cos\delta \\ \operatorname{cosec}\alpha & \operatorname{cosec}\beta & \operatorname{cosec}\delta \end{pmatrix}$ is a Square Matrix of order 3

(ii) ROW MATRIX

A matrix having only one row is called Row Matrix

For example

A = $(2 \quad 0 \quad 1)$ is a row matrix of order 1 x 3

B = $(1 \quad 0)$ is a row matrix or order 1 x 2

(iii) COLUMN MATRIX

A matrix having only one column is called Column Matrix.

For example

$A = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ is a column matrix of order 3 x 1 and

$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a column matrix of order 2 x 1

(iv) ZERO OR NULL MATRIX

A matrix in which all elements are equal to zero is called Zero or Null Matrix and is denoted by O.

For example

$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a Null Matrix of order 2 x 2 and

$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a Null Matrix of order 2 x 3

(v) DIAGONAL MATRIX

A square Matrix in which all the elements other than main diagonal elements are zero is called a diagonal matrix

For example

$A = \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix}$ is a Diagonal Matrix of order 2 and

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a Diagonal Matrix of order 3

Consider the square matrix

$$A = \begin{pmatrix} 1 & 3 & 7 \\ 5 & -2 & -4 \\ 3 & 6 & 5 \end{pmatrix}$$

Here 1, -2, 5 are called main diagonal elements and 3, -2, 7 are called secondary diagonal elements.

(vi) SCALAR MATRIX

A Diagonal Matrix with all diagonal elements equal to K (a scalar) is called a Scalar Matrix.

For example

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ is a Scalar Matrix of order 3 and the value of scalar K = 2

(vii) UNIT MATRIX OR IDENTITY MATRIX

A scalar Matrix having each diagonal element equal to 1 (unity) is called a Unit Matrix and is denoted by I.

For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ is a Unit Matrix of order 2

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is a Unit Matrix of order 3

1.1.4 Multiplication of a marix by a scalar

If $A = (a_{ij})$ is a matrix of any order and if K is a scalar, then the Scalar Multiplication of A by the scalar k is defined as

$$KA = (Ka_{ij}) \text{ for all i, j.}$$

In other words, to multiply a matrix A by a scalar K, multiply every element of A by K.

1.1.5 Negative of a matrix

The negative of a matrix $A = (a_{ij})_{m \times n}$ is defined by $-A = (-a_{ij})_{m \times n}$ for all i, j and is obtained by changing the sign of every element.

For example

If $A = \begin{pmatrix} 2 & -5 & 7 \\ 0 & 5 & 6 \end{pmatrix}$ then

$-A = \begin{pmatrix} -2 & 5 & -7 \\ 0 & -5 & -6 \end{pmatrix}$

1.1.6 Equality of matrices

Two matrices are said to equal when

i) they have the same order and
ii) the corresponding elements are equal.

1.1.7 Addition of matrices

Addition of matrices is possible only when they are of same order (i.e., conformal for addition). When two matrices A and B are of same order, then their sum (A+B) is obtained by adding the corresponding elements in both the matrices.

1.1.8 Properties of matrix addition

Let A, B, C be matrices of the same order. The addition of matrices obeys the following

(i) Commutative law : A + B = B + A
(ii) Associative law : A + (B + C) = (A + B) + C
(iii) Distributive law : K(A+B) = KA+KB, where k is scalar.

1.1.9 Subtraction of matrices

Subtraction of matrices is also possible only when they are of same order. Let A and B be the two matrices of the same order. The matrix A - B is obtained by subtracting the elements of B from the corresponding elements of A.

1.1.10 Multiplication of matrices

Multiplication of two matrices is possible only when the number of columns of the first matrix is equal to the number of rows of the second matrix (i.e. conformable for multiplication)

Let A $= (a_{ij})$ be an m x p matrix,
and let B $= (b_{ij})$ be an p x n matrix.

Then the product AB is a matrix C = (c_{ij}) of order mxn,

where c_{ij} = element in the i^{th} row and j^{th} column of C is found by multiplying corresponding elements of the i^{th} row of A and j^{th} column of B and then adding the results.

For example

$$\text{if } A = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 6 & 7 \end{pmatrix}_{3\text{x}2} \quad B = \begin{pmatrix} 5 & -7 \\ -2 & 4 \end{pmatrix}_{2\text{x}2}$$

$$\text{then } AB = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 6 & 7 \end{pmatrix} \quad \begin{pmatrix} 5 & -7 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3\text{ x }5 + 5\text{x}(-2) & 3\text{x}(-7) + 5\text{x}(5) \\ 2\text{ x }5 + (-1)\text{ x }(-2) & 2\text{ x }(-7) + (-1)\text{ x }(4) \\ 6\text{ x }5 + 7\text{x }(-2) & 6\text{x }(-7) + 7\text{x }(4) \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 12 & -18 \\ 16 & -14 \end{pmatrix}$$

1.1.11 Properties of matrix multiplication

(i) Matrix Multiplication is not commutative i.e. for the two matrices A and B, generally AB $\neq$ BA.

(ii) The Multiplication of Matrices is associative
i.e., (AB) C = A(BC)

(iii) Matrix Multiplication is distributive with respect to addition. i.e. if, A, B, C are matrices of order mxn, n x k, and n x k respectively, then A(B+C) = AB + AC

(iv) Let A be a square matrix of order n and I is the unit matrix of same order.
Then AI = A = I A

(v) The product
AB = O (Null matrix), does not imply that either A = 0 or B = 0 or both are zero.

For example

$$\text{Let } A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}_{2 \times 2} \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}_{2 \times 2}$$

$$\text{Then } AB = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ AB = (null matrix)

Here neither the matrix A, nor the matrix B is Zero, but the product AB is zero.

1.1.12 Transpose of a matrix

Let A = (a_{ij}) be a matrix of order mxn. The transpose of A, denoted by A^T of order nxm is obtained by interchanging rows into columns of A.

For example

$$\text{If} \quad A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}_{2 \times 3} \text{, then}$$

$$A^T = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$$

1.1.13 Properties Of Matrix Transposition

Let A^T and B^T are the transposed Matrices of A and B and α is a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(\alpha A)^T = \alpha A^T$

(iv) $(AB)^T = B^T A^T$ (A and B are conformable for multiplication)

Example 1

$$\textbf{If } A = \begin{pmatrix} 5 & 9 & 6 \\ 6 & 2 & 10 \end{pmatrix} \textbf{ and } B = \begin{pmatrix} 6 & 0 & 7 \\ 4 & -8 & -3 \end{pmatrix}$$

find A + B and A-B

Solution :

$$A+B = \begin{pmatrix} 5+6 & 9+0 & 6+7 \\ 6+4 & 2+(-8) & 10+(-3) \end{pmatrix} = \begin{pmatrix} 11 & 9 & 13 \\ 10 & -6 & 7 \end{pmatrix}$$

$$A-B = \begin{pmatrix} 5-6 & 9-0 & 6-7 \\ 6-4 & 2-(-8) & 10-(-3) \end{pmatrix} = \begin{pmatrix} -1 & 9 & -1 \\ 2 & 10 & 13 \end{pmatrix}$$

Example 2

If A = $\begin{pmatrix} \mathbf{3} & \mathbf{6} \\ \mathbf{9} & \mathbf{2} \end{pmatrix}$ **find (i) 3A (ii) -** $\frac{\mathbf{1}}{\mathbf{3}}$ **A**

Solution :

$$\text{(i) } 3A = 3\begin{pmatrix} 3 & 6 \\ 9 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 18 \\ 27 & 6 \end{pmatrix}$$

$$\text{(ii) } -\frac{1}{3}A = -\frac{1}{3}\begin{pmatrix} 3 & 6 \\ 9 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -3 & -\frac{2}{3} \end{pmatrix}$$

Example 3

If A = $\begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{5} \\ \mathbf{4} & \mathbf{7} & \mathbf{9} \\ \mathbf{1} & \mathbf{6} & \mathbf{4} \end{pmatrix}$ **and B =** $\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{2} & \mathbf{5} \\ \mathbf{6} & \mathbf{-2} & \mathbf{7} \end{pmatrix}$

show that 5(A+B) = 5A + 5B

Solution :

$$A+B = \begin{pmatrix} 5 & 4 & 7 \\ 8 & 9 & 14 \\ 7 & 4 & 11 \end{pmatrix} \quad \therefore 5(A+B) = \begin{pmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{pmatrix}$$

$$5A = \begin{pmatrix} 10 & 15 & 25 \\ 20 & 35 & 45 \\ 5 & 30 & 20 \end{pmatrix} \text{ and } 5B = \begin{pmatrix} 15 & 5 & 10 \\ 20 & 10 & 25 \\ 30 & -10 & 35 \end{pmatrix}$$

$$\therefore 5A+5B = \begin{pmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{pmatrix} \quad \therefore 5(A+B) = 5A + 5B$$

Example 4

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ **and** $\mathbf{B} = \begin{pmatrix} -1 & -2 & -4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}$

find AB and BA. Also show that AB ¹ BA

Solution:

$$AB \quad = \begin{pmatrix} 1(-1)+2(-1)+3(1) & 1(-2)+2(-2)+3x2 & 1(-4)+2(-4)+3x4 \\ 2(-1)+4(-1)+6(1) & 2(-2)+4(-2)+6(2) & 2(-4)+4(-4)+6x4 \\ 3(-1)+6(-1)+9(1) & 3(-2)+6(-2)+9(2) & 3(-4)+6(-4)+9x4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3x\,3}$$

$$\text{Similarly } BA = \begin{pmatrix} -17 & -34 & -51 \\ -17 & -34 & -51 \\ 17 & 34 & 51 \end{pmatrix}$$

$$\therefore \; AB \neq BA$$

Example 5

If $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$ **, then compute** $\mathbf{A^2\text{-}5A + 3I}$

Solution:

$$A^2 = A.A \; = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} -5 & 6 \\ -9 & 10 \end{pmatrix}$$

$$5A \; = 5\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 5 & -10 \\ 15 & -20 \end{pmatrix}$$

$$3I \; = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\therefore \; A^2 \text{ - } 5A+3I \quad = \begin{pmatrix} -5 & 6 \\ -9 & 10 \end{pmatrix} - \begin{pmatrix} 5 & -10 \\ 15 & -20 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 16 \\ -24 & 30 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 16 \\ -24 & 33 \end{pmatrix}$$

Example 6

Verify that $(AB)^T = B^T A^T$ when

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & 2 \\ 4 & 0 & 1 \end{pmatrix}_{2\times3} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -3 \\ 0 & 1 \\ -4 & -2 \end{pmatrix}_{3\times2}$$

Solution :

$$AB = \begin{pmatrix} 1 & -4 & 2 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 0 & 1 \\ -4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1x2 + (-4)x0 + 2(-4) & 1x(-3) + (-4)x1 + 2x(-2) \\ 4x2 + 0x0 + 1x(-4) & 4x(-3) + 0x1 + 1x(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2+0-8 & -3-4-4 \\ 8+0-4 & -12+0-2 \end{pmatrix} = \begin{pmatrix} -6 & -11 \\ 4 & -14 \end{pmatrix}$$

$$\therefore \text{L.H.S.} = (AB)^T = \begin{pmatrix} -6 & -11 \\ 4 & -14 \end{pmatrix}^T = \begin{pmatrix} -6 & 4 \\ -11 & -14 \end{pmatrix}$$

$$\text{R.H.S.} = B^T A^T = \begin{pmatrix} 2 & 0 & -4 \\ -3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ -4 & 0 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 4 \\ -11 & -14 \end{pmatrix}$$

$\Rightarrow$ L.H.S. = R.H.S

Example 7

A radio manufacturing company produces three models of radios say A, B and C. There is an export order of 500 for model A, 1000 for model B, and 200 for model C. The material and labour (in appropriate units) needed to produce each model is given by the following table:

	Material	Labour
Model A	**10**	**20**
Model B	**8**	**5**
Model C	**12**	**9**

Use marix multiplication to compute the total amount of material and labour needed to fill the entire export order.

Solution:

Let P denote the matrix expressing material and labour corresponding to the models A, B, C. Then

$$P = \begin{pmatrix} \overset{\text{Material}}{10} & \overset{\text{Labour}}{20} \\ 8 & 05 \\ 12 & 9 \end{pmatrix} \begin{matrix} \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{matrix}$$

Let E denote matrix expressing the number of units ordered for export in respect of models A, B, C. Then

$$E = \begin{pmatrix} \overset{A}{500} & \overset{B}{1000} & \overset{C}{200} \end{pmatrix}$$

$\therefore$ Total amount of material and labour = E x P

$$= (500 \quad 1000 \quad 200) \begin{pmatrix} 10 & 20 \\ 8 & 5 \\ 12 & 9 \end{pmatrix}$$

$$= (5000 + 8000 + 2400 \qquad 10000 + 5000 + 1800)$$

$$= \begin{pmatrix} \overset{\text{Material}}{15,400} & \overset{\text{Labour}}{16,800} \end{pmatrix}$$

Example 8

Two shops A and B have in stock the following brand of tubelights

Shops	Brand		
	Bajaj	**Philips**	**Surya**
Shop A	**43**	**62**	**36**
Shop B	**24**	**18**	**60**

Shop A places order for 30 Bajaj, 30 Philips, and 20 Surya brand of tubelights, whereas shop B orders 10, 6, 40 numbers of the three varieties. Due to the various factors, they receive only half of the order as supplied by the manufacturers. The cost of each tubelights of the three types are Rs. 42, Rs. 38 and Rs. 36 respectively. Represent the following as matrices (i) Initial stock (ii) the order (iii) the supply (iv) final sotck (v) cost of individual items (column matrix) (vi) total cost of stock in the shops.

Solution:

(i) The initial stock matrix $P = \begin{pmatrix} 43 & 62 & 36 \\ 24 & 18 & 60 \end{pmatrix}$

(ii) The order matrix $Q = \begin{pmatrix} 30 & 30 & 20 \\ 10 & 6 & 40 \end{pmatrix}$

(iii) The supply matrix $R = \frac{1}{2} Q = \begin{pmatrix} 15 & 15 & 10 \\ 5 & 3 & 20 \end{pmatrix}$

(iv) The final stock matrix $S = P + R = \begin{pmatrix} 58 & 77 & 46 \\ 29 & 21 & 80 \end{pmatrix}$

(v) The cost vector $C = \begin{pmatrix} 42 \\ 38 \\ 36 \end{pmatrix}$

(vi) The total cost stock in the shops

$$T = SC = \begin{pmatrix} 58 & 77 & 46 \\ 29 & 21 & 80 \end{pmatrix} \begin{pmatrix} 42 \\ 38 \\ 36 \end{pmatrix}$$

$$= \begin{pmatrix} 2436 + 2926 + 1656 \\ 1218 + 798 + 2880 \end{pmatrix} = \begin{pmatrix} 7018 \\ 4896 \end{pmatrix}$$

EXERCISE 1.1

1) If $A = \begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 4 & 6 \end{pmatrix}$ then, show that

(i) $A + B = B + A$ (ii) $(A^T)^T = A$

2. If $A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 9 & 8 \\ 2 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & 2 & 5 \\ 0 & 3 & -1 \\ 4 & -6 & 2 \end{pmatrix}$

find (i) A + B (iii) 5A and 2B

(ii) B + A (iv) 5A + 2B

3) If $A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -3 & 0 \end{pmatrix}$, find AB and BA.

4) Find AB and BA when

$$A = \begin{pmatrix} -3 & 1 & -5 \\ -1 & 5 & 2 \\ -2 & 4 & -3 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 4 & 5 \\ 0 & 2 & 1 \\ -1 & 6 & 3 \end{pmatrix}$$

5) If $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 \\ 7 & 3 \\ 5 & -2 \end{pmatrix}$, find AB and BA.

6) If $A = \begin{pmatrix} 3 & 4 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$

verify that $(AB)^T = B^T A^T$

7) Let $A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -5 \end{pmatrix}$ then

show that 3 (A+B) = 3A + 3B.

8) If $A = \begin{pmatrix} 12 & 11 \\ 9 & -7 \end{pmatrix}$, $\alpha = 3$, $\beta = -7$,

show that $(\alpha + \beta)A = \alpha A + \beta A$.

9) Verify that $\alpha (A + B) = \alpha A + \alpha B$ where

$$\alpha = 3, \quad A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \\ 4 & 3 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 3 & -1 \\ 7 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

10) If $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and $B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

prove that (i) AB = BA (ii) $(A+B)^2 = A^2 + B^2 + 2AB$.

11) If $A = (3 \ 5 \ 6)_{1 \times 3}$, and $B = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}_{3 \times 1}$ then find AB and BA.

12) If $A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$ find AB, BA

13) There are two families A and B. There are 4 men, 2 women and 1 child in family A and 2 men, 3 women and 2 children in family B. They recommended daily allowance for calories i.e. Men : 2000, Women : 1500, Children : 1200 and for proteins is Men : 50 gms., Women : 45 gms., Children : 30 gms.

Represent the above information by matrices using matrix multiplication, calculate the total requirements of calories and proteins for each of the families.

14) Find the sum of the following matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 7 & 10 & 12 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 9 & 7 \\ 7 & 8 & 6 \\ 9 & 10 & 8 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 7 & 13 & 19 \end{pmatrix}$$

15) If $x + \begin{pmatrix} 5 & 6 \\ 7 & 0 \end{pmatrix} = 2I_2 + 0_2$ then find x

16) If $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ show that $(A - I)(A - 4I) = 0$

17) If $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ then show that

(i) $(A+B)(A-B) \neq A^2 - B^2$ (ii) $(A+B)^2 \neq A^2 + 2AB + B^2$

18) If $3A + \begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix}$, find the value of A

19) Show that $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfies $A^2 = -I$

20) If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ prove that $A^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

21) If $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$ show that A^2, A^4 are identity matrices

22) If $A = \begin{pmatrix} 7 & 1 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix}$

Evluate (i) (A+B) (C+D) (ii) (C+D) (A+B) (iii) $A^2 - B^2$ (iv) $C^2 + D^2$

23) The number of students studying Business Mathematics, Economics, Computer Science and Statistics in a school are given below

Std.	Business Mathematics	Economics	Computer Science	Statistics
XI Std.	45	60	55	30
XII Std.	58	72	40	80

(i) Express the above data in the form of a matrix

(ii) Write the order of the matrix

(iii) Express standardwise the number of students as a column matrix and subjectwise as a row matrix.

(iv) What is the relationship between (i) and (iii)?

1.2 DETERMINANTS

An important attribute in the study of Matrix Algebra is the concept of **Determinant,** ascribed to a square matrix. A knowledge of **Determinant** theory is indispensable in the study of Matrix Algebra.

1.2.1 Determinant

The determinant associated with each square matrix $A = (a_{ij})$ is a **scalar** and denoted by the symbol det.A or $|A|$. The scalar may be real or complex number, positive, Negative or Zero. A matrix is an array and has**no numerical** value, but a determinant has **numerical value.**

For example

when $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then determinant of A is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$ and the determinant value is = ad - bc

Example 9

Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-2} \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}$$

= 1 x (-2) - 3 x (-1) = -2 + 3 = 1

Example 10

Evaluate $\begin{vmatrix} \mathbf{2} & \mathbf{0} & \mathbf{4} \\ \mathbf{5} & \mathbf{-1} & \mathbf{1} \\ \mathbf{9} & \mathbf{7} & \mathbf{8} \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 0 & 4 \\ 5 & -1 & 1 \\ 9 & 7 & 8 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 7 & 8 \end{vmatrix} -0\begin{vmatrix} 5 & 1 \\ 9 & 8 \end{vmatrix} + 4\begin{vmatrix} 5 & -1 \\ 9 & 7 \end{vmatrix}$$

= 2 (-1 x 8 - 1 x 7) - 0 (5 x 8 -9 x 1) + 4 (5x7 - (-1) x 9)

= 2 (-8 -7) - 0 (40 - 9) + 4 (35 + 9)

= -30 - 0 + 176 = 146

1.2.2 Properties Of Determinants

(i) The value of determinant is unaltered, when its rows and columns are interchanged.

(ii) If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes only in sign.

(iii) If the determinant has two identical rows (columns), then the value of the determinant is zero.

(iv) If all the elements in a row or in a (column) of a determinant are multiplied by a constant k(k, $\neq$ 0) then the value of the determinant is multiplied by k.

(v) The value of the determinant is unaltered when a constant multiple of the elements of any row (column), is added to the corresponding elements of a different row (column) in a determinant.

(vi) If each element of a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant is expressed as the sum of two or more determinants of the same order.

(vii) If any two rows or columns of a determinant are proportional, then the value of the determinant is zero.

1.2.3 Product of Determinants

Product of two determinants is possible only when they are of the same order. Also |AB| = |A| . |B|

Example 11

Evaluate $|A|\ |B|$, if $A = \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix}$ **and** $B = \begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix}$

Solution:

Multiplying row by column

$$|A|\ |B| = \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} \begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 \text{ x } 5 + 1 \text{ x } 1 & 3 \text{ x } 2 + 1 \text{ x } 3 \\ 5 \text{ x } 5 + 6 \text{ x } 1 & 5 \text{ x } 2 + 6 \text{ x } 3 \end{vmatrix}$$

$$= \begin{vmatrix} 15+1 & 6+3 \\ 25+6 & 10+18 \end{vmatrix} = \begin{vmatrix} 16 & 9 \\ 31 & 28 \end{vmatrix} = 448 - 279$$

$$= 169$$

Example 12

Find $\begin{vmatrix} 2 & 1 & 3 \\ 3 & 0 & 5 \\ 1 & 0 & -4 \end{vmatrix} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix}$

Solution :

Multiplying row by column

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 0 & 5 \\ 1 & 0 & -4 \end{vmatrix} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 \text{ x } 2 + 1 \text{ x } 0 + 3 \text{ x } 0 & 2 \text{ x } 0 + 1 \text{ x } 0 + 3 \text{ x } 2 & 2 \text{ x } 0 + 1 \text{ x } 3 + 3 \text{ x } 0 \\ 3 \text{ x } 2 + 0 \text{ x } 0 + 5 \text{ x } 0 & 3 \text{ x } 0 + 0 \text{ x } 0 + 5 \text{ x } 2 & 3 \text{ x } 0 + 0 \text{ x } 3 + 5 \text{ x } 0 \\ 1 \text{ x } 2 + 0 \text{ x } 0 - 4 \text{ x } 0 & 1 \text{ x } 0 + 0 \text{ x } 0 - 4 \text{ x } 2 & 1 \text{ x } 0 + 0 \text{ x } 3 - 4 \text{ x } 0 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 6 & 3 \\ 6 & 10 & 0 \\ 2 & -8 & 0 \end{vmatrix}$$

= 4 (0 + 0) -6 (0 - 0) + 3 (-48 - 20)

= 3 (-68) = - 204

1.2.4 Singular Matrix

A square matrix A is said to be singular if det. A = 0, otherwise it is a non-singular matrix.

Example 13

Show that $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$ **is a singular matrix**

Solution:

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$

$\therefore$ The matrix is singular

Example 14

Show that $\begin{pmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{9} & \mathbf{10} \end{pmatrix}$ **is a non-singular matrix**

Solution :

$\begin{vmatrix} 2 & 5 \\ 9 & 10 \end{vmatrix} = 29 - 45 = -25 \neq 0$

$\therefore$ The given matrix is non singular

Example : 15

Find x if $\begin{vmatrix} 1 & x & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{vmatrix} = 0$

Solution :

Expanding by 1st Row,

$$\begin{vmatrix} 1 & x & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ -4 & 8 \end{vmatrix} - x\begin{vmatrix} 5 & 0 \\ -2 & 8 \end{vmatrix} + (-4)\begin{vmatrix} 5 & 3 \\ -2 & -4 \end{vmatrix}$$

$= 1(24) - x(40) - 4(-20 + 6)$

$= 24 - 40x + 56 = -40x + 80$

$\Rightarrow -40x + 80 = 0$

$\therefore x = 2$

Example : 16

Show $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$ **= (a-b) (b-c) (c-a)**

Solution :

$$\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & a^2+b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & (a+b)(a-b) \\ 0 & a-c & (a+c)(a-c) \end{vmatrix}$$ taking out (a-b) from R_2 and (a-c) from R_3

$$= (a-b)(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

= (a-b) (a-c) [a+c-a-b] (Expanding along c_1)

= (a-b) (a-c) (c-b) = (a-b) (b-c) (c-a)

EXERCISE 1.2

1) Evaluate (i) $\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$ (iii) $\begin{vmatrix} -2 & -4 \\ -1 & -6 \end{vmatrix}$

2) Evaluate $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 1 & 2 & 4 \end{vmatrix}$ 3) Evaluate $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

4) Examine whether $A = \begin{pmatrix} 7 & 4 & 3 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \end{pmatrix}$ is non-singular

5) Examine whether the given matrix $A = \begin{pmatrix} 1 & -2 & 3 \\ -2 & -1 & 0 \\ 4 & -2 & 5 \end{pmatrix}$ is singular

6) Evaluate $\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{vmatrix}$ 7) Evaluate $\begin{vmatrix} 1 & 4 & 2 \\ 2 & -2 & 4 \\ 3 & -1 & 6 \end{vmatrix}$

8) If the value of $\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix}$ = -60, then evaluate $\begin{vmatrix} 2 & 6 & 5 \\ 4 & 2 & 0 \\ 6 & 4 & 7 \end{vmatrix}$

9) If the value of $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{vmatrix}$ = 5, then what is the value of $\begin{vmatrix} 1 & 8 & 3 \\ 1 & 7 & 3 \\ 2 & 12 & 1 \end{vmatrix}$

10) Show that $\begin{vmatrix} 2+4 & 6+3 \\ 1 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 6 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 1 & 5 \end{vmatrix}$

11) Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

12) Prove that $\begin{vmatrix} b+c & a & 1 \\ c+a & b & 1 \\ a+b & c & 1 \end{vmatrix} = 0$

13) Show that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

EXERCISE 1.3

Choose the correct answer

1) [0 0 0] is a

(a) Unit matrix (b) Scalar matrix
(c) Null matrix (d) Diagonal matrix

2) [6 2 - 3] is a matrix of order

(a) 3 x 3 (b) 3 x 1
(c) 1 x 3 (d) Scalar matrix

3) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a

(a) Unit matrix (b) Zero matrix of order 2 x 2
(c) Unit matrix of 2 x 2 (c) None of these

4) $A = \begin{pmatrix} 3 & -3 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, then A + B is

(a) $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5) If A = $\begin{pmatrix} 8 & 9 \\ -3 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 3 \\ 0 & -2 \end{pmatrix}$, then A - B is

(a) $\begin{pmatrix} 7 & 6 \\ -3 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} 9 & 6 \\ -3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & 6 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6) If A = $\begin{pmatrix} 2 & 4 \\ -3 & -3 \end{pmatrix}$, then -3A is

(a) $\begin{pmatrix} -6 & -12 \\ -9 & 15 \end{pmatrix}$ (b) $\begin{pmatrix} -6 & -12 \\ 9 & 15 \end{pmatrix}$ (c) $\begin{pmatrix} -6 & 12 \\ 9 & 9 \end{pmatrix}$ (d) None of these

7) If A = $\begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 5 & -3 & 1 \end{pmatrix}$ and I = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then A + 2I is

(a) $\begin{pmatrix} 4 & 3 & 4 \\ 1 & 1 & 0 \\ 5 & -3 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 3 & 4 \\ 1 & 0 & 0 \\ 5 & -3 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 3 & 4 \\ 1 & -1 & 0 \\ 5 & -3 & 2 \end{pmatrix}$ (d) None of these

8) $\begin{pmatrix} 3 & 5 & 6 \\ -2 & 1 & 6 \end{pmatrix}$ x $\begin{pmatrix} 5 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

a) $\begin{pmatrix} 15 & 12 \\ -4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 15 \\ 8 & -3 \end{pmatrix}$

(c) Cannot be multiplied (d) None of these

9) The value of $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ is

(a) 4 (b) 14 (c) -14 (d) None of these

10) The value of $\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$ is

(a) 0 (b) - 1 (c) 1 (d) None of these

11) If the value of $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ = -2, then the value of $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$ is

(a) 0 (b) - 2 (c) 2 (d) None of these

12) Det (AB) = | AB | = ?

(a) | A | + | B | (b) | B | + | A |

(c) | A | x | B | (d) None of these

13) The element at 2nd Row and 2nd Coloumn is denoted by

(a) a_{12} (b) a_{32} (c) a_{22} (d) a_{11}

14) Order of the matrix A = $[a_{ij}]_{3 \times 3}$ is

(a) 2 x 3 (b) 3 x 3 (c) 1 x 3 (d) 3 x 1

15) When the number of rows and the number of coloumns of a matrix are equal, the matrix is

(a) square matrix (b) row matrix (c) column matrix (d) None of these

16) If all the elements of a matrix are zeros, then the matrix is a

(a) unit matrix (b) square matrix

(c) zero matrix (d) None of these

17) A diagonal matrix in which all the diagonal elements are equal is a

(a) scalar matrix (b) column matrix

(c) unit matrix (d) None of these

18) If any two rows and coloumns of a determinant are identical, the value of the determinant is

(a) 1 (b) 0 (c) -1 (d) unaltered

19) If there is only one column in a matrix, it is called

(a) Row matrix (b) column matrix

(c) square matrix (d) rectangular

20) Addition of matrices is

(a) not commutative (b) commutative

(c) not associative (d) distributive

21) A square matrix A is said to be non-singular if

(a) $|A| \neq 0$ (b) $|A| = 0$ (c) A = 0 (d) None of these

22) The value of x if $\begin{vmatrix} 1 & x \\ 5 & 3 \end{vmatrix} = 0$ is

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) 0 (d) None of these

23) If $\begin{vmatrix} 4 & 8 \\ -9 & 4 \end{vmatrix} = 88$, then the value of $\begin{vmatrix} 8 & 4 \\ 4 & -9 \end{vmatrix}$ is

(a) -88 (b) 88 (c) 80 (d) None of these

24) The value of $\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix}$ is

(a) 0 (b) -1 (c) 1 (d) None of these

25) If $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2$, then the value of $\begin{vmatrix} 2 & 6 \\ 2 & 4 \end{vmatrix}$ is

(a) -2 (b) 2 (c) -4 (d) None of these

26) If (A+B) (A-B) = $A^2 - B^2$ and A and B are square matrices then

(a) $(AB)^T = AB$ (b) AB = BA

(c) (A+B)T = $B^T + A^T$ (d) None of these

27) $\begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix}$ is a

(a) Rectangular matrix (b) Scalar matrix

(c) Identity matrix (d) None of these

28) $\begin{pmatrix} 1 \\ 2 \\ 6 \\ 7 \end{pmatrix}$ is a

(a) Square matrix (b) Row matrix

(c) Scalar matrix (d) Column matrix

29) If A = I, then A^2

(a) I^2 (b) I (c) 0 (d) None of these

30) If A = (1 2 3) and B = $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then the order of AB is

(a) 1 x 1 (b) 1 x 3 (c) 3 x 1 (d) 3 x 3

ALGEBRA 2

2.1 PARTIAL FRACTION

We know that two or more rational expressions of the form p/q can be added and subtracted. In this chapter we are going to learn the process of writing a single rational expression as a sum or difference of two or more rational expressions. This process is called splitting up into partial fractions.

(i) Every rational expression of the form p/q where q is the non-repeated product of linear factors like (ax+b) (cx+d), can be represented as a partial fraction of the form: $\frac{M}{ax+b} + \frac{N}{cx+d}$, where M and N are the constants to be determined.

For example: $\frac{2x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3}$, where A and B are to be determined.

(ii) Every rational expression of the form p/q, where q is linear expression of the type (ax+b) occurring in multiples say n times i.e., $(ax+b)^n$ can be represented as a partial fraction of the form:

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \ldots\ldots + \frac{A_n}{(ax+b)^n}$$

For example : $\frac{1}{(x-1)(x-2)^2} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$

(iii) Every rational expression of the form p/q where q is an irreducible quadratic expression of the type ax^2+bx+c, can be equated to a partial fraction of the type

$$\frac{Ax+B}{ax^2+bx+c}$$

For example : $\frac{2x+7}{(3x^2+5x+1)(4x+3)} = \frac{Ax+B}{3x^2+5x+1} + \frac{C}{4x+3}$

Example 1

Resolve into partial fractions $\frac{4x+1}{(x-2)(x+1)}$

Solution:

Step 1: Let $\frac{4x+1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$ ------------------(1)

Step 2: Taking L.C.M. on R.H.S.

$$\frac{4x+1}{(x-2)(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

Step 3: Equating the numerator on both sides

$4x+1 = A(x+1) + B(x-2)$

$= Ax+A + Bx-2B$

$= (A+B)x + (A-2B)$

Step 4: Equating the coefficient of like terms,

$A+B = 4$ -----------------(2)

$A-2B = 1$ -----------------(3)

Step 5: Solving the equations (2) and (3) we get

$A = 3$ and $B = 1$

Step 6: Substituting the values of A and B in step 1 we get

$$\frac{4x+1}{(x-2)(x+1)} = \frac{3}{x-2} + \frac{1}{x+1}$$

Example 2

Resolve into partial fractions $\frac{1}{(x-1)(x+2)^2}$

Solution:

Step 1: Let $\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

Step 2: Taking L.C.M. on R.H.S we get

$$\frac{1}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$$

Step 3: Equating Numerator on either sides we get

$1 = A(x+2)^2+B(x-1)(x+2)+C(x-1)$

Step 4: Puting $x = -2$ we get $C = -\frac{1}{3}$

Step 5: Puting x = 1 we get $A = \frac{1}{9}$

Step 6: Putting x = 0 and substituting the values of A and C in step 3 we get

$B = -\frac{1}{9}$

Step 7: $\therefore \frac{1}{(x-1)(x+2)^2} = \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$

Example 3

Resolve into partial fractions $\frac{x^2+1}{x(x+1)^2}$

Solution:

Step 1: Let $\frac{x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

Step 2: Taking L.C.M. on R.H.S. we get

$$\frac{x^2+1}{x(x+1)^2} = \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}$$

Step 3: Equating the Numerator on either sides we get
$x^2+1 = A(x+1)^2 + Bx(x+1) + Cx$

Step 4: Putting x = 0 we get A = 1

Step 5: Putting x = -1 we get C = -2

Step 6: Putting x = 2 and substituting the values of A and C in step 3 we get B = 0

Step 7: $\therefore \frac{x^2+1}{x(x+1)^2} = \frac{1}{x} + \frac{0}{x+1} - \frac{2}{(x+1)^2} = \frac{1}{x} - \frac{2}{(x+1)^2}$

Example 4

Resolve into partial fractions $\frac{x^2-2x-9}{(x^2+x+6)(x+1)}$

Solution:

Step 1: Let $\frac{x^2-2x-9}{(x^2+x+6)(x+1)} = \frac{Ax+B}{x^2+x+6} + \frac{C}{x+1}$

($\because$ x^2+x+6 cannot be factorised)

Step 2: Taking L.C.M. on R.H.S. we get

$$\frac{x^2-2x-9}{(x^2+x+6)(x+1)} = \frac{(Ax+B)(x+1)+C(x^2+x+6)}{(x^2+x+6)(x+1)}$$

Step 3: Equating the Numerator on either side we get
$x^2-2x-9 = (Ax+B)(x+1)+C(x^2+x+6)$

Step 4: Putting $x = -1$ we get $C = -1$

Step 5: Putting $x = 0$ and substituting the value of C we get $B = -3$

Step 6: Putting $x = 1$ and substituting the values of B and C in step 3 get $A = 2$

Step 7: $\therefore \frac{x^2-2x-9}{(x^2+x+6)(x+1)} = \frac{2x-3}{x^2+x+6} - \frac{1}{x+1}$

Example 5

Resolve into partial fraction $\frac{1}{(x^2+4)(x+1)}$

Solution:

Step 1: Let $\frac{1}{(x^2+4)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$

Step 2: Taking L.C.M. on R.H.S. we get

$$\frac{1}{(x^2+4)(x+1)} = \frac{A(x^2+4)+(Bx+c)(x+1)}{(x+1)(x^2+4)}$$

Step 3: Equating the Numerator on either side we get
$1 = A(x^2+4) + (Bx + C)(x + 1)$

Step 4: Putting $x = -1$ we get $A = \frac{1}{5}$

Step 5: Putting $x = 0$ and substituting the value of A we get $C = \frac{1}{5}$

Step 6: Putting $x = 1$ and substituting the value of A and C in Step 3 we get $B = -\frac{1}{5}$

Step 7: $\therefore \frac{1}{(x^2+4)(x+1)} = \frac{1}{5(x+1)} + \frac{\frac{-1}{5}x + \frac{1}{5}}{x^2+4}$

EXERCISE 2.1

Resolve into partial fractions

1) $\frac{x+1}{x^2-x-6}$ 2) $\frac{2x-15}{x^2+5x+6}$

3) $\frac{1}{x^2-1}$ 4) $\frac{x+4}{(x^2-4)(x+1)}$

5) $\frac{x+1}{(x-2)^2(x+3)}$ 6) $\frac{1}{(x-1)(x+2)^2}$

7) $\frac{x}{(x-1)(x+1)^2}$ 8) $\frac{2x^2+7x+23}{(x-1)(x+3)^2}$

9) $\frac{7x^2-25x+6}{(x^2-2x-1)(3x-2)}$ 10) $\frac{x+2}{(x-1)(x^2+1)}$

2.2 PERMUTATIONS

This topic deals with the new Mathematical idea of counting without doing actual counting. That is without listing out particular cases it is possible to assess the number of cases under certain given conditions.

Permutations refer to different arrangement of things from a given lot taken one or more at a time. For example, Permutations made out of a set of three elements {a,b,c}

(i) One at a time: {a}, {b}, {c} ...... 3 ways

(ii) Two at a time: {a,b}, {b,a},{b,c}, {c,b}, {a,c}, {c,a} ...... 6 ways

(iii) Three at a time: {a,b,c}, {a,c,b}, {b,c,a}, {b,a,c}, {c,a,b}, {c,b,a} ......6 ways

2.2.1 Fundamental rules of counting

There are two fundamental rules of counting based on the simple principles of multiplication and addition, the former when events occur independently one after another and latter when either of the events can occur simultaneously. Some times we have to combine the two depending on the nature of the problem.

2.2.2 Fundamental principle of counting

Let us consider an example from our day-to-day life. Sekar was allotted a roll number for his examination. But he forgot his number. What all he remembered was that it was a two digit odd number.

The possible numbers are listed as follows:

11	21	31	41	51	61	71	81	91
13	23	33	43	53	63	73	83	93
15	25	35	45	55	65	75	85	95
17	27	37	47	57	67	77	87	97
19	29	39	49	59	69	79	89	99

So the total number of possible two digit odd numbers = 9x5 = 45

Let us see whether there is any other method to find the total number of two digit odd numbers. Now the digit in the unit place can be any one of the five digits 1,3,5,7,9. This is because our number is an odd number. The digit in the ten's place can be any one of the nine digits 1,2,3,4,5,6,7,8,9

Thus there are five ways to fill up the unit place and nine ways to fill up the ten's place. So the total number of two digit odd numbers = 9x5 = 45. This example illustrates the following principle.

(i) Multiplication principle

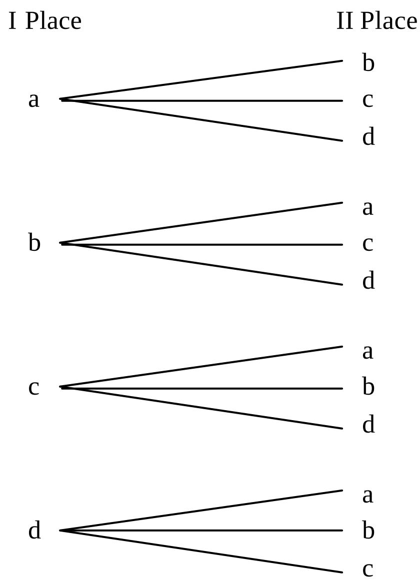
If one operation can be performed in "m" different ways and another operation can be performed in "n" different ways then the two operations together can be performed in 'm x n' different ways. This principle is known as *multiplication principle* of counting.

(ii) Addition Principle

If one operation can be performed in *m* ways and another operation can be performed in *n* ways, then any one of the two operations can be performed in m+n ways. This principle known as *addition principle* of counting.

Further consider the set {a,b,c,d}

From the above set we have to select two elements and we have to arrange them as follows.



The possible arrangements are

(a,b), (a,c), (a,d)
(b,a), (b,c), (b,d)
(c,a), (c,b), (c,d)
(d,a), (d,b), (d,c)

The total number of arrangements are 4 x 3 = 12

In the above arrangement, the pair (a,b) is different from the pair (b,a) and so on. There are 12 possible ways of arranging the letters a,b,c,d taking two at a time.

i.e Selecting and arranging ‘2’ from ‘4’ can be done in 12 ways. In otherwords number of permutations of ‘four’ things taken ‘two’ at a time is 4x3 = 12

In general $^{n}p_{r}$ denotes the number of permutations of ‘n’ things taken ‘r’ at a time.

[‘n’ and ‘r’ are positive integers and $r \leq n$]

2.2.3 To find the value of ^np_r:

^np_r means selecting and arranging 'r' things from 'n' things which is the same as filling 'r' places using 'n' things which can be done as follows.

The first place can be filled by using anyone of 'n' things in 'n' ways

The second place can be filled by using any one of the remaining (n-1) things in (n-1) ways.

So the first and the second places together can be filled in n(n-1) ways.

The third place can be filled in (n-2) ways by using the remaining (n-2) things.

So the first, second and the third places together can be filled in n(n-1) (n-2) ways.

In general 'r' places can be filled in n(n-1)(n-2)....[n-(r-1)] ways.

So $^np_r = n(n-1)(n-2)...(n-r+1)$. To simplify the above formula, we are going to introduce factorial notation.

2.2.4 Factorial notation:

The product of first 'n' natural numbers is called n- factorial denoted by n ! or $\underline{|n}$.

For example:

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$
$$4! = 4 \times 3 \times 2 \times 1$$
$$\therefore 5! = 5 \times 4!$$
$$5! = 5 \times 4 \times 3!$$

In general, $n! = n(n-1)(n-2)...3.2.1$

$$\therefore n! = n\{(n-1)!\}$$
$$= n(n-1)(n-2)! \text{ and so on}$$

We have $^np_r = n(n-1)(n-2)............(n-r+1)$

$$= \frac{n(n-1)(n-2)......(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

{multiplying and dividing by (n-r)!}

$$\therefore \; {}^np_r = \frac{n!}{(n-r)!}$$

Observation :

(i) o! = 1

(ii) $^{n}p_{o} = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

(iii) $^{n}p_{1} = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$

(iv) $^{n}p_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

(ie. Selecting and arranging 'n' things from 'n' things can be done in n! ways).

(i.e 'n' things can be arranged among themselves in n! ways).

2.2.5 Permutations of repeated things:

If there are 'n' things of which 'm' are of one kind and the remaining (n-m) are of another kind, then the total number of distinct permutations of 'n' things

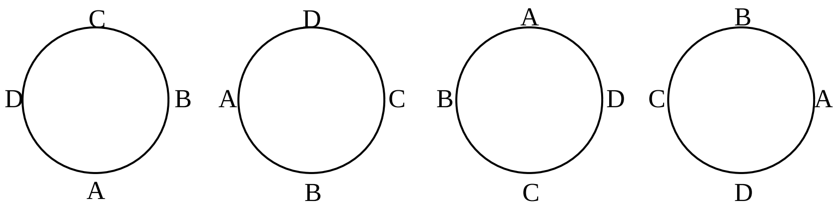
$$= \frac{n!}{m!(n-m)!}$$

If there are m_1 things of first kind, m_2 things of second kind and m_r things of r th kind such that $m_1+m_2+.....+m_r = n$ then the total number of permutations of 'n' things

$$= \frac{n!}{m_1!m_2!......m_r!}$$

2.2.6 Circular Permutations:

We have seen permutations of 'n' things in a row. Now we consider the permutations of 'n' things in a circle. Consider four letters A,B,C,D. The four letters can be arranged in a row in 4! ways. Of the 4! arrangements, the arrangement ABCD, BCDA, CDAB, DABC are the same when represented along a circle.



So the number of permutations of '4' things along a circle is $\frac{4!}{4} = 3!$

In general, n things can be arranged among themselves in a circle in (n-1)! ways

Example 6

Find the value of (i) $^{10}p_1$, **(ii)** $^{7}p_4$, **(iii)** $^{11}p_0$

Solution:

i) $^{10}p_1 = 10$

ii) $^{7}p_4 = \frac{\underline{|7}}{\underline{|7-4}} = \frac{\underline{|7}}{\underline{|3}} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 7 \times 6 \times 5 \times 4 = 840$

iii) $^{11}p_0 = 1$

Example 7

There are 4 trains from Chennai to Madurai and back to Chennai. In how many ways can a person go from Chennai to Madurai and return in a different train?

Solution:

Number of ways of selecting a train from Chennai to Madurai from the four trains = $^{4}p_1$ = 4ways

Number of ways of selecting a train from Madurai to Chennai from the remaining 3 trains = $^{3}p_1$ = 3 ways

$\therefore$ Total number of ways of making the journey = 4 x 3 = 12ways

Example 8

There is a letter lock with 3 rings each marked with 4 letters and do not know the key word. How many maximum useless attempts may be made to open the lock?

Solution:

To open the lock :
The number of ways in which the first ring's position can be fixed using the four letters = $^{4}p_1$ = 4 ways

The number of ways in which the second ring's position can be fixed using the 4 letters = $^{4}p_1$ = 4 ways

The number of ways in which the third ring's position can be fixed using the 4 letters = $^4p_1$ = 4 ways

$\therefore$ Total number of attempts = 4 x 4 x 4 = 64 ways

Of these attempts, only one attempt will open the lock.

$\therefore$ Maximum number of useless attempts = 64 - 1 = 63

Example 9

How many number of 4 digits can be formed out of the digits 0,1,2,..........,9 if repetition of digits is not allowed.

Solution:

The number of ways in which the 1000's place can be filled (0 cannot be in the 1000's place) = 9ways

The number of ways in which the 100's place 10's place and the unit place filled using the remaining 9 digits (including zero) = $^9p_3$ = 504 ways

$\therefore$ Total number of 4 digit numbers formed = 9 x 504 = 4536

Example 10

Find the number of arrangements of 6 boys and 4 girls in a line so that no two girls sit together

Solution:

Six boys can be arranged among themselves in a line in 6! ways. After this arrangement we have to arrange the four girls in such a way that in between two girls there is atleast one boy. So the possible places to fill with the girls are as follows

☐ B ☐ B ☐ B ☐ B ☐ B ☐ B ☐

The four girls can be arranged in the boxes (7 places) which can be done in $^7p_4$ ways. So the total number of arrangements = 6! x $^7p_4$ = 720 x 7 x 6 x 5 x 4 = 604800

Example 11

A family of 4 brothers and 3 sisters are to be arranged in a row. In how many ways can they be seated if all the sisters sit together?

Solution:

Consider the 3 sisters as one unit. There are 4 brothers which is treated as 4 units. Now there are totally 5 units which can be arranged among themselves in 5! ways. After these arrangements the 3 sisters can be arranged among themselves in 3! ways.

$\therefore$ Total number of arrangement = 5! x 3! = 720

Example 12

Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.

Solution:

Number of 4 digit numbers that can be formed using the digits 2, 3, 4, 5 is ${}^4p_4$=4! = 24. Out of the 24 numbers the digit 2 appears in the unit place 6 times, the digit 3 appears in the unit place 6 times and so on. If we write all the 24 numbers and add, the sum of all the numbers in the unit place

= 6[2+3+4+5] = 6 x 14 = 84

Similarly the sum of all the numbers in the 10's place = 84
The sum of all the numbers in the 100's place = 84
and the sum of all the numbers in the 1000's place = 84
$\therefore$ sum of all the 4 digit numbers = 84x1000 + 84x100+84x10+84x1
= 84 (1000+100+10+1) = 84 x 1111
= 93324

Example 13

In how many ways can the letters of the word CONTAMINATION be arranged?

Solution:

The number of letters of word CONTAMINATION = 13
which can be arranged in 13! ways
Of these arrangements the letter O occurs 2 times
N occurs 3 times
T occurs 2 times
A occurs 2 times
and I occurs 2 times

$\therefore$ The total number of permutations = $\dfrac{13!}{2!\,3!\,2!\,2!\,2!}$

EXERCISE 2.2

1) If ${}^{n}p_5 = (42)\,{}^{n}p_3$, find n

2) If $6[{}^{n}p_3] = 7\,{}^{(n-1)}p_3$ find n

3) How many distinct words can be formed using all the letters of the word
i) ENTERTAINMENT ii) MATHEMATICS iii) MISSISSIPPI

4) How many even numbers of 4 digits can be formed out of the digits 1,2,3,....9 if repetition of digits is not allowed?

5) Find the sum of all numbers that can be formed with the digits 3,4,5,6,7 taken all at a time.

6) In how many ways can 7 boys and 4 girls can be arranged in a row so that
i) all the girls sit together ii) no two girls sit together?

7) In how many ways can the letters of the word STRANGE be arranged so that vowels may appear in the odd places.

8) In how many ways 5 gentlemen and 3 ladies can be arranged along a round table so that no two ladies are together?

9) Find the number of words that can be formed by considering all possible permutations of the letters of the word FATHER. How many of these words begin with F and end with R?

2.3 COMBINATIONS

Combination are selections ie. it inolves only the selection of the required number of things out of the total number of things. Thus in combination order does not matter.

For example, consider a set of three elements {a,b,c} and combination made out of the set with

i) One at a time: {a}, {b}, {c}
ii) Two at a time: {a,b}, {b,c}, {c,a}
iii) Three at a time: {a,b,c}

The number of comibnations of n things taken r, $(r \leq n)$ is denoted by ${}^{n}c_r$ or $\binom{n}{r}$

2.3.1 To derive the formula for ^nc_r:

Number of combinations of 'n' things taken 'r' at a time = ^nc_r
Number of permutations of 'n' things taken 'r' at a time = ^np_r
Number of ways 'r' things can be arranged among themselves = r!
Each combination having r things gives rise to r! permutations

$\therefore \quad ^np_r = (^nc_r)\, r!$

$=> \quad \dfrac{n!}{(n-r)!} = (^nc_r)\, r!$

$\therefore \quad ^nc_r = \dfrac{n!}{r!(n-r)!}$

Observation:

(i) $^nc_o = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{n!} = 1$

(ii) $^nc_n = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{r!0!} = 1$

(iii) $^nc_r = {}^nc_{n-r}$

(iv) If $^nc_x = {}^nc_y$ then x = y or x+y = n

(v) $^nc_r = \dfrac{^np_r}{r!}$

Example14

Evaluate $^8p_3$ and $^8c_3$

Solution:

$^8p_3 = \dfrac{8!}{(8-3)!} = \dfrac{8!}{5!} = \dfrac{8x7x6x5!}{5!} = 8 \text{ x } 7 \text{ x } 6 = 336$

$^8c_3 = \dfrac{8!}{3!(8-3)!} = \dfrac{8!}{3!\ 5!} = \dfrac{8x7x6x5!}{3!5!} = \dfrac{8x7x6}{3x2x1} = 56$

Example 15

Evaluate $^{10}c_8$

Solution:

$^{10}c_8 = {}^{10}c_2 = \dfrac{10x9}{2x1} = 45$

Example 16

If $^{n}c_8 = {}^{n}c_6$, find $^{n}c_2$.

Solution:

$^{n}c_8 = {}^{n}c_6$ (given)

=> n = 8+6 = 14

$\therefore \quad {}^{n}c_2 = {}^{14}c_2 = \frac{14 \times 13}{2 \times 1} = 91$

Example 17

If $\binom{100}{r} = \binom{100}{4r}$, find 'r'

Solution:

$^{100}c_r = {}^{100}c_{4r}$ (given)

=> r + 4r = 100

$\therefore$ r = 20

Example 18

Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels.

Solution:

Selecting 3 from 7 consonants can be done in $^{7}c_3$ ways

Selecting 2 from 4 vowels can be done in $^{4}c_2$ ways.

$\therefore$ Total number of words formed = $^{7}c_3 \times {}^{4}c_2$

$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1}$

$\therefore$ = 35 x 6 = 210

Example 19

There are 13 persons in a party. If each of them shakes hands with each other, how many handshakes happen in the party?

Solution:

Selecting two persons from 13 persons can be done in $^{13}c_2$ ways.

$\therefore$ Total number of hand shakes = $^{13}c_2 = \frac{13 \times 12}{2 \times 1} = 78$

Example 20

There are 10 points in a plane in which none of the 3 points are collinear. Find the number of lines that can be drawn using the 10 points.

Solution:

To draw a line we need atleast two points. Now selecting 2 from 10 can be done in $^{10}c_2$ ways

$\therefore$ number of lines drawn = $^{10}c_2 = \frac{10 \times 9}{2 \times 1} = 45$

Example 21

A question paper has two parts, part A and part B each with 10 questions. If the student has to choose 8 from part A and 5 from part B, in how many ways can he choose the questions?

Solution:

Number of questions in part A = 10.
Selecting 8 from part A can be done in $^{10}c_8$ ways = $^{10}c_2$

Number of questions in part B = 10
Selecting 5 from part B can be done in $^{10}c_5$ ways

$\therefore$ Total number of ways in which the questions can be selected

$= {}^{10}c_8 \times {}^{10}c_5 = 45 \times 252 = 11340$ ways

Example 22

A committee of seven students is formed selecting from 6 boys and 5 girls such that majority are from boys. How many different committees can be formed?

Solution:

Number of students in the committee = 7
Number of boys = 6
Number of girls = 5
The selection can be done as follows

Boy (6)	Girl (5)
6	1
5	2
4	3

ie. (6B and 1G) or (5B and 2G) or (4B and 3G)

The possible ways are $\binom{6}{6}\binom{5}{1}$ or $\binom{6}{5}\binom{5}{2}$ or $\binom{6}{4}\binom{5}{3}$

$\therefore$ The total number of different committees formed

$$= {}^6c_6 \times {}^5c_1 + {}^6c_5 \times {}^5c_2 + {}^6c_4 \times {}^5c_3$$

$$= 1 \times 5 + 6 \times 10 + 15 \times 10 = 215$$

2.3.2 Pascal's Triangle

For n = 0, 1, 2, 3, 4, 5 ... the details can be arranged in the form of a triangle known as Pascal's triangle.

n = 0 $\binom{0}{0}$

n = 1 $\binom{1}{0}$ $\binom{1}{1}$

n = 2 $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

n = 3 $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

n = 4 $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

n = 5 $\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

Substituting the values we get

n = 0 1

n = 1 1 1

n = 2 1 2 1

n = 3 1 3 3 1

n = 4 1 4 6 4 1

n = 5 1 5 10 10 5 1

The conclusion arrived at from this triangle named after the French Mathematician Pascal is as follows. The value of any entry in any row is equal to sum of the values of the two entries in the preceding row on either side of it. Hence we get the result.

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

2.3.3 Using the formula for ^nc_r derive that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$

Proof :

L.H.S. $= {}^nc_r + {}^nc_{r-1}$

$$= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1!)[n-(r-1)]!}$$

$$= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1!)(n-r+1)!}$$

$$= \frac{n!\,[n-r+1] + n!\,(r)}{r!\,(n+1-r)!}$$

$$= \frac{n!\,[n-r+1+r]}{r!\,(n-r+1)!} = \frac{n!(n+1)}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!}$$

$= {}^{n+1}c_r =$ R.H.S.

EXERCISE 2.3

1) Evaluate a) $^{10}c_6$ b) $^{15}c_{13}$

2) If $^{36}c_n = {}^{36}c_{n+4}$, find 'n'.

3) $^{n+2}c_n = 45$, find n.

4) A candidate is required to answer 7 questions out of 12 questions which are divided into two groups each containing 6 questions. He is not permitted to attempt more than 5 questions from each group. In how many ways can he choose the 7 questions.

5) From a set of 9 ladies and 8 gentlemen a group of 5 is to be formed. In how many ways the group can be formed so that it contains majority of ladies

6) From a class of 15 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen.

7) Find the number of diagonals of a hexagon.

8) A cricket team of 11 players is to be chosen from 20 players including 6 bowlers and 3 wicket keepers. In how many different ways can a team be formed so that the team contains exactly 2 wicket keepers and atleast 4 bowlers.

2.4 MATHEMATICAL INDUCTION

Many mathematical theorems, formulae which cannot be easily derived by direct proof are sometimes proved by the indirect method known as mathematical induction. It consists of three steps.

(i) Actual verification of the theorem for n = 1

(ii) Assuming that the theorem is true for some positive integer k(k>1). We have to prove that the theorem is true for k+1 which is the integer next to k.

(iii) The conclusion is that the theorem is true for all natural numbers.

2.4.1 Principle of Mathematical Induction:

Let P(n) be the statement for $n \in N$. If P(1) is true and P(k+1) is also true whenever P(k) is true for k > 1 then P(n) is true for all natural numbers.

Example 23

Using the principle of Mathematical Induction prove that for all $n \in N$, $1+2+3+...n = \frac{n(n+1)}{2}$

Solution:

Let $P(n) = \frac{n(n+1)}{2}$

For L.H.S. n=1, p(1) = 1

For R.H.S $p(1) = \frac{1(1+1)}{2} = 1$

L.H.S = R.H.S for n = 1

$\therefore$ P(1) is true.

Now assume that P(k) is true

i.e. $1+2+3+......+k = \frac{k(k+1)}{2}$ is true.

To prove that p(k+1) is true

Now **p(k+1) = p(k) + t_{k+1}**

p(k+1) = 1+2+3+......k+k+1

= p(k) + (k+1)

$= \frac{k(k+1)}{2} + k+1$

$= (k+1)\left[\frac{k}{2}+1\right]$

$= \frac{(k+1)(k+2)}{2}$

=> p(k+1) is true whenever p(k) is true. But p(1) is true.

$\therefore$ p(n) is true for all $n \in N$.

Example 24

Show by principle of mathematical induction that $3^{2n} - 1$ is divisible by 8 for all $n \in N$.

Solution:

Let P(n) be the given statement

p(1) $= 3^2 - 1 = 9-1 = 8$ which is divisible by 8.

$\therefore$ p(1) is true.

Assume that p(k) is true

ie., $3^{2k}-1$ is divisible by 8.

To prove p(k+1) is true.

Now p(k+1) $= 3^{2(k+1)} - 1 = 3^{2k} \times 3^2 - 1$

$= 9\ 3^{2k} - 1$

$= 9(3^{2k}) - 9 + 8$

$= 9[3^{2k}-1] + 8$

Which is divisible by 8 as $3^{2k}-1$ is divisible by 8

So p(k+1) is true whenever p(k) is true. So by induction p(n) is true for all $n \in N$.

EXERCISE 2.4

By the principle of mathematical induction prove the following

1) $1+3+5+\ldots\ldots (2k-1) = k^2$

2) $4+8+12+\ldots\ldots.4n = 2n(n+1)$

3) $1.2 + 2.3 + 3.4 + \ldots\ldots n(n+1) = \frac{n(n+1)(n+2)}{3}$

4) $1^3 + 2^3 + \ldots\ldots n^3 = \frac{n^2(n+1)^2}{4}$

5) $1^2 + 2^2 + \ldots\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$

6) $1+4+7+10+\ldots\ldots(3n-2) = \frac{n}{2}(3n-1)$

7) $2^{3n} - 1$ is divisible by 7.

2.4.2 Summation of Series

We have $1+2+3+\ldots\ldots+n = \Sigma n = \frac{n(n+1)}{2}$

$$1^2+2^2+\ldots\ldots+n^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+\ldots\ldots+n^3 = \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Thus $\boxed{\mathbf{Sn = \frac{n(n+1)}{2}}}$

$$\boxed{\mathbf{Sn^2 = \frac{n(n+1)(2n+1)}{6}}}$$

$$\boxed{\mathbf{Sn^3 = \left\{\frac{n(n+1)}{2}\right\}^2}}$$

Using the above formula we are going to find the summation when the nth term of the sequence is given.

Example 25

Find the sum to n terms of the series whose nth term is n(n+1)(n+4)

Solution :

$t_n = n(n+1)(n+4)$

$\quad = n^3 + 5n^2 + 4n$

$\therefore S_n = \Sigma t_n = \Sigma(n^3 + 5n^2 + 4n)$

$\quad = \Sigma n^3 + 5\,\Sigma n^2 + 4\Sigma n$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 5\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 4\left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)}{12} [3n^2 + 23n + 34]$$

Example 26

Sum to n terms of the series $1^2.3 + 2^2.5 + 3^2.7 + ...$

Solution:

The n^{th} term is $n^2(2n+1) = 2n^3+n^2$

$$\therefore S_n = \Sigma(2n^3+n^2) = 2\Sigma n^3 + \Sigma n^2$$

$$= \frac{2n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2} [n(n+1) + \frac{2n+1}{3}]$$

$$= \frac{n(n+1)}{2} \left(\frac{3n^2+3n+2n+1}{3}\right)$$

$$= \frac{n(n+1)}{6} [3n^2 + 5n + 1]$$

Example 27

Sum the following series 2+5+10+17+......to n terms

Solution:

2+5+10+17+.......

= (1+1) + (1+4) + (1+9) + (1+16)+......

= (1+1+1+.......n terms) + $(1^2+2^2+.....n^2)$

$$= n+\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n}{6} [6+2n^2+3n+1]$$

$$= \frac{n}{6} [2n^2+3n+7]$$

EXERCISE 2.5

Find the sum to n terms of the following series

1) 1.2.3 + 2.3.4 + 3.4.5 + ...

2) $1.2^2 + 2.3^2 + 3.4^2 + ...$

3) $2^2 + 4^2 + 6^2 + ...(2n)^2$

4) 2.5 + 5.8 + 8.11 + ...

5) $1^2 + 3^2 + 5^2 + ...$

6) 1 + (1+2) + (1+2+3) + ...

2.5 BINOMIAL THEOREM

2.5.1 Theorem

If n is a natural number,

$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + ... + {}^nC_r\, x^{n-r} a^r + ... {}^nC_n\, a^n$

Proof:

We shall prove the theorem by the principle of Mathematical Induction

Let P(n) denote the statement :

$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + ...$
$+ {}^nC_{r-1}\, x^{n+1-r} a^{r-1} + {}^nC_r\, x^{n-r} a^r + ... + {}^nC_n\, a^n$

Let n = 1, Then LHS of P(1) = x + a

RHS of P(1) = 1 . x + 1 . a = x + a = L.H.S. of P (1)

∴ P (1) is true

Let us assume that the statement P (k) be true for $k \in N$

i.e. P(k) :

$(x+a)^k = {}^kC_0\, x^k + {}^kC_1\, x^{k-1} a + {}^kC_2\, x^{k-2} a^2 + ...$
$+ {}^kC_{r-1}\, x^{k+1-r} a^{r-1} + {}^kC_r\, x^{k-r} a^r + ... + {}^kC_k\, a^k$ .... (1)

is true

To prove P (k+1) is true

i.e., $(x+a)^{k+1} = {}^{k+1}C_0\, x^{k+1} + {}^{k+1}C_1\, x^k a$
$+ {}^{k+1}C_2\, x^{k-1} a^2 + ... + {}^{k+1}C_r\, x^{k+1-r} a^r + ... + ... + {}^{k+1}C_{k+1}\, a^{k+1}$ is true.

$(x+a)^{k+1} = (x+a)\,(x+a)^k$
$= (x + a)\,[{}^kC_0\, x^k + {}^kC_1\, x^{k-1} a + {}^kC_2\, x^{k-2} a^2 + ... + {}^kC_{r-1}\, x^{k+1-r} a^{r-1}$
$+ {}^kC_r\, x^{k-r} a^r + ... + {}^kC_k\, a^k]$ using (1)

$= {}^{k}C_0\, x^{k+1} + {}^{k}C_1\, x^k a + {}^{k}C_2\, x^{k-1} a^2 + \ldots + {}^{k}C_r\, x^{k+1-r} a^r + \ldots + {}^{k}C_k\, x\, a^k$

$+ {}^{k}C_0\, x^k a + {}^{k}C_1\, x^{k-1} a + \ldots {}^{k}C_{r-1}\, x^{k+1-r} a^r + \ldots + {}^{k}C_k\, a^{k+1}$

$= {}^{k}C_0\, x^{k+1} + ({}^{k}C_1 + {}^{k}C_0)\, x^k a + ({}^{k}C_2 + {}^{k}C_1)\, x^{k-1} a^2 + \ldots\ldots..$

$\ldots\ldots.. + ({}^{k}C_r + {}^{k}C_{r-1})\, x^{k+1-r} a^r + \ldots + {}^{k}C_k\, a^{k+1}$

But ${}^{k}C_r + {}^{k}C_{r-1} = {}^{k+1}C_r$

Put r = 1, 2, ... etc.

${}^{k}C_1 + {}^{k}C_0 = {}^{k+1}C_1,\ {}^{k}C_2 + {}^{k}C_1 = {}^{k+1}C_2$ ......

${}^{k}C_0 = 1 = {}^{k+1}C_0\ ;\ {}^{k}C_k = 1 = {}^{k+1}C_{k+1}$

$\therefore (x+a)^{k+1} = {}^{k+1}C_0\, x^{k+1} + {}^{k+1}C_1\, x^k a + {}^{k+2}C_2\, x^{k-1} a^2 + \ldots\ldots$

$+ {}^{k+1}C_r\, x^{k+1-r} a^r + \ldots\ldots + {}^{k+1}C_{k+1}\, a^{k+1}$

Thus if P (k) is true, then P (k +1) is also true.

$\therefore$ By the principle of mathematical induction P(n) is true for $n \in N$.

Thus the Binomial Theorem is proved for $n \in N$.

Observations:

(i) The expansion of $(x+a)^n$ has (n+1) terms.

(ii) The general term is given by $t_{r+1} = nC_r\, x^{n-r} a^r$.

(iii) In $(x+a)^n$, the power of 'x' decreases while the power of 'a' increases such that the sum of the indices in each term is equal to n.

(iv) The coefficients of terms equidistant from the beginning and end are equal.

(v) The expansion of $(x+a)^n$ has (n+1) terms Let n+1 = N.

a) when N is odd the middle term is $t_{\frac{N+1}{2}}$

b) when N is even the middle terms are $t_{\frac{N}{2}}$ and $t_{\frac{N}{2}+1}$

(vi) Binomial cooefficients can also be represented by C_0, C_1, C_2, etc.

2.5.2 Binomial coefficients and their properties

$(1+x)^n = C_0 + C_1 x + C_2\, x^2 + C_3\, x^3 + \ldots.. + C_n x^n$ ....(1)

Put x = 1 in (1) we get

$2^n = C_0 + C_1 + C_2 + \ldots\ldots. + C_n$

Put x = -1 in (1) we get

$0 = C_0 - C_1 + C_2 - C_3 + \ldots.. + (-1)^n C_n$

$\Rightarrow C_0 + C_2 + C_4 + \ldots\ldots. = C_1 + C_3 + \ldots.$

=> sum of the coefficients of even terms = $\frac{2^n}{2} = 2^{n-1}$

sum of the coefficients of odd terms = 2^{n-1}

Example 28

Expand $(x+\frac{1}{x})^4$

Solution :

$(x+\frac{1}{x})^4 = 4C_0\, x^4 + 4C_1\, x^3(\frac{1}{x}) + 4C_2\, x^2(\frac{1}{x})^2 + 4C_3\, x(\frac{1}{x})^3 + 4C_4(\frac{1}{x})^4$

$= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

Example 29

Expand $(x+3y)^4$

Solution :

$(x+3y)^4 = 4C_0\, x^4 + 4C_1\, x^3(3y) + 4C_2\, x^2\,(3y)^2 + 4C_3\, x(3y)^3 + 4C_4(3y)^4$

$= x^4 + 4x^3(3y) + 6x^2(9y^2) + 4x(27y^3) + 81y^4$

$= x^4 + 12x^3y + 54x^2y^2 + 108xy^3 + 81y^4$

Example 30

Find the 5th term of $(2x-3y)^7$

Solution :

$t_{r+1} = 7C_r(2x)^{7-r}\,(-3y)^r$

$\therefore \quad t_5 = t_{4+1} = 7C_4\,(2x)^{7-4}\,(-3y)^4$

$= 7C_3\,(2x)^3\,(3y)^4$

$= \frac{7 \text{ x } 6 \text{ x } 5}{3 \text{ x } 2 \text{ x } 1}\,(8x^3)\,(81y^4)$

$= (35)\,(8x^3)\,(81y^4) = 22680x^3y^4$

Example 31

Find the middle term(s) in the expansion of $(x-\frac{2}{x})^{11}$

Solution :

$n = 11$

$\therefore$ $n+1 = 12 = N$ = even number

So middle terms = $t_{\frac{N}{2}}$ and $t_{(\frac{N}{2}+1)}$

ie., t_6 and t_7

(i) Now $t_6 = t_{5+1}$ $= 11C_5\, x^{11-5} (-\frac{2}{x})^5$

$= 11C_5\, x^6 \frac{(-2)^5}{x^5}$

$= -11C_5 \frac{x^6 2^5}{x^5}$

$= -11C_5\, 2^5 x = (-11C_5)(32x)$

(ii) $t_7 = t_{6+1} = 11C_6 (x)^{11-6} (-\frac{2}{x})^6$

$= 11C_6\, x^5 \frac{(-2)^6}{x^6}$

$= 11C_6 \frac{x^5 2^6}{x^6}$

$= 11C_6 \left(\frac{64}{x}\right)$

Example 32

Find the coefficient of x^{10} in the expansion of $(2x^2 - \frac{3}{x})^{11}$

Solution :

General term $= t_{r+1} = 11C_r(2x^2)^{11-r} (-\frac{3}{x})^r$

$= 11C_r\, 2^{11-r} (x^2)^{11-r} \frac{(-3)^r}{x^r}$

$= 11C_r\, 2^{11-r} x^{22-2r} (-3)^r x^{-r}$

$= 11C_r\, 2^{11-r} (-3)^r x^{22-3r}$

To find the coefficient of x^{10}, the index of x must be equated to 10.

$\Rightarrow 22-3r = 10$

$22-10 = 3r$

$\therefore r = 4$

So coefficient of x^{10} is $11C_4\, 2^{11-4} (-3)^4 = 11C_4 (2^7) (3^4)$

Example 33

Find the term independent of x in the expansion of $(\frac{4x^2}{3} - \frac{3}{2x})^9$

Solution :

General term $= t_{r+1} = 9C_r(\frac{4x^2}{3})^{9-r}(\frac{-3}{2x})^r$

$= 9C_r \frac{4^{9-r}}{3^{9-r}} \times \frac{(-3)^r}{2^r} \times (x^2)^{9-r} \frac{1}{x^r}$

$= 9C_r \frac{4^{9-r}}{3^{9-r}} \times \frac{(-3)^r}{2^r} x^{18-2r} x^{-r}$

$= 9C_r \frac{4^{9-r}}{3^{9-r}} \frac{(-3)^r}{2^r} x^{18-3r}$

The term independent of x = constant term = coefficient of x^0

∴ To find the term independent of x

The power of x must be equated to zero

=> $18-3r = 0$

∴ $r = 6$

So the term independent of x is $9C_6 \frac{4^{9-6}}{3^{9-6}} \frac{(-3)^6}{2^6}$

$= 9C_3 \frac{4^3}{3^3} \frac{(3)^6}{(2)^6}$

$= \frac{9x8x7}{3x2x1} \times \frac{64}{3^3} \times \frac{3^6}{64}$

$= (84)(3^3) = 84x27 = 2268$

EXERCISE 2.6

1) Find the middle term(s) in the expansion of $(x-\frac{2}{x})^{11}$

2) Find the coefficient of x^{-8} in the expansion of $(x--\frac{2}{x})^{20}$

3) Find the term independent of x in the expansion of $(x^2-\frac{4}{x^3})^{10}$

4) Find the 8th term in the expansion of $(2x+\frac{1}{y})^9$

5) Find the middle term in the expansion of $(3x - \frac{x^3}{6})^9$

6) Find the term independent of x in the expansion of $(2x^2 + \frac{1}{x})^{12}$

7) Show that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1.3.5\ldots(2n-1)\ 2^n . x^n}{n!}$$

8) Show that the middle term in the expansion of $(x + \frac{1}{2x})^{2n}$ is $\frac{1.3.5\ldots(2n-1)}{n!}$

EXERCISE 2.7

Choose the correct answer

1) If n! = 24 then n is
(a) 4 (b) 3 (c) 4! (d) 1

2) The value of 3! + 2! + 1! + 0! is
(a) 10 (b) 6 (c) 7 (d) 9

3) The value of $\frac{1}{4!} + \frac{1}{3!}$ is
(a) $\frac{5}{20}$ (b) $\frac{5}{24}$ (c) $\frac{7}{12}$ (d) $\frac{1}{7}$

4) The total number of ways of analysing 6 persons around a table is
(a) 6 (b) 5 (c) 6! d) 5!

5) The value of x(x-1) (x-2)! is
(a) x! (b) (x-1)! (c) (x-2)! (d) (x+1)!

6) 2 persons can occupy 7 places in ____ ways
(a) 42 (b) 14 (c) 21 (d) 7

7) The value of $^8p_3$ is
(a) 8 x 7 x 6 (b) $\frac{8 \times 7 \times 6}{3 \times 2 \times 1}$ (c) 8 x 7 (d) 3 x 2 1

8) The value of $^8C_0$ is
(a) 8 (b) 1 (c) 7 (d) 0

9) The value of $^{10}C_9$ is
(a) 9 (b) 1 (c) $^{10}C_1$ (d) 0

10) Number of lines that can be drawn using 5 points in which none of 3 points are collinear is
(a) 10 (b) 20 (c) 5 (d) 1

11) If $\binom{5}{x} + \binom{5}{4} = \binom{6}{5}$ then x is

(a) 5 (b) 4 (c) 6 (d) 0

12) If $^{10}c_r = {}^{10}c_{4r}$ then r is

(a) 2 (b) 4 (c) 10 (d) 1

13) Sum of all the binomial coefficients is

(a) 2^n (b) b^n (c) 2n (d) n

14) The last term in $(x+1)^n$ is

(a) x^n (b) b^n (c) n (d) 1

15) The number of terms in $(2x+5)^7$ is

(a) 2 (b) 7 (c) 8 (d) 14

16) The middle term in $(x+a)^8$ is

(a) t_4 (b) t_5 (c) t_6 (d) t_3

17) The general term in $(x+a)^n$ is denoted by

(a) t_n (b) t_r (c) t_{r-1} (d) t_{r+1}

SEQUENCES AND SERIES 3

A *sequence* is defined as a function from the set of natural numbers N or a subset of it to the set of real numbers R. The domain of a sequence is N or a subset of N and the codomain is R.

We use the notation t_n to denote the image of the natural number n. We use $\{t_n\}$ or $<t_n>$ to describe a sequence. Also $t_1, t_2, t_3, \ldots$ are called the terms of the sequence. The distinctive terms of a sequence constitute its range. A sequence with finite number of terms is called a finite sequence. A sequence with infinite number of terms is an infinite sequence.

Examples of finite sequences are

(i) $t_n = \frac{n}{n+3}$, $n < 10$

The domain of the sequence is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

and the range is $\{\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \frac{6}{9}, \frac{7}{10}, \frac{8}{11}, \frac{9}{12}\}$

(ii) $t_n = 2+(-1)^n$

The domain is $\{1, 2, 3, \ldots \}$

The range is $\{1, 3\}$

Examples of infinite sequences are

(i) t_n = the n^{th} prime number

(ii) t_n = the integral part of $+\sqrt{n}$

It is not necessary that terms of a sequence follow a definite pattern or rule. The general term need not be capable of being explicitly expressed by a formula. If the terms follow a definite rule then the sequence is called a *progression*. All progressions are sequences but all sequences need not be progressions. Examples of progressions are

(i) 5, 10, 15, 20, 25,...

(ii) 1, -1, 1, -1, 1, ...

(iii) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, ...

(iv) 1, 1, 2, 3, 5, 8, 13, ...

(v) 2, 6, 3, 9, 4, 12, ...etc.

The algebraic sum of the terms of a sequence is called a series. Thus $\frac{3}{2} + \frac{5}{3} + \frac{7}{4} + \ldots$ is the series corresponding to the sequence $\frac{3}{2}$, $\frac{5}{3}$, $\frac{7}{4}$, ...

We shall study sequences in their general form in sequel. Now we recall two progressions.

(i) Arithmetic Progression (A.P.)

(ii) Geometric Progression (G.P.)

Arithmetic Progression (A.P.)

A sequence is said to be in A.P. if its terms continuously increase or decrease by a fixed number. The fixed number is called the common difference of the A.P.

The standard form of an A.P. may be taken as a, a+d, a+2d, a+3d,...
Here the first term is 'a' and the common difference is 'd'

The n^{th} term or the general term of the A.P. is $t_n = a + (n-1)\,d$.

The sum to n terms of the A.P. is $S = \frac{n}{2}\,[2a + (n-1)\,d]$

If three numbers a, b, c are in A.P. then $b = \frac{a+c}{2}$

Geometric Progression (G.P.)

A sequence is said to be in G.P. if every term bears to the preceding term a constant ratio. The constant ratio is called the common ratio of the G.P.

The standard form of a G.P. may be taken as a, ar, ar^2, ar^3,...

Here the first term is 'a' and the common ratio is 'r'. The n^{th} term or the general term of the G.P. is $t_n = ar^{n-1}$

The sum to n terms of the G.P. is $S = a\dfrac{(1-r^n)}{1-r}$

If three numbers a, b, c are in G.P. then $b^2 = ac$.

3.1 HARMONIC PROGRESSION (H.P.)

The receiprocals of the terms of an A.P. form an H.P.

Thus if $a_1, a_2, a_3, ..., a_n,...$ are in A.P. then $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, ..., \dfrac{1}{a_n}, ...$ form an H.P.

Suppose a, b, c be in H.P. Then $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ will be in A.P.

$\therefore \quad \dfrac{1}{b} = \dfrac{\frac{1}{a}+\frac{1}{c}}{2}$ i.e. $b = \dfrac{2ac}{a+c}$

Example 1

Find the seventh term of the H.P. $\dfrac{1}{5}, \dfrac{1}{9}, \dfrac{1}{13}, ...$

Solution:

Consider the associated A.P., 5, 9, 13, ...

$t_n = a + (n-1)\, d$

$t_7 = 5 + (7-1)\, 4 = 29$

$\therefore$ the seventh term of the given H.P. is $\dfrac{1}{29}$

Example 2

If a, b, c be in H.P., prove that $\dfrac{b+a}{b-a} + \dfrac{b+c}{b-c} = 2$

Solution:

Given that a, b, c are in H.P.

$\therefore \quad b = \dfrac{2ac}{a+c}$ -----------------(1)

i.e. $\frac{b}{a} = \frac{2c}{a+c}$

Applying componendo et dividendo,

$$\frac{b+a}{b-a} = \frac{2c+a+c}{2c-a-c}$$

i.e. $\frac{b+a}{b-a} = \frac{3c+a}{c-a}$ -----------------(2)

Again from (1)

$$\frac{b}{c} = \frac{2a}{a+c}$$

Applying componendo et dividendo,

$$\frac{b+c}{b-c} = \frac{2a+a+c}{2a-a-c}$$

i.e. $\frac{b+a}{b-c} = \frac{3a+c}{a-c}$ ----------------(3)

Adding (2) and (3)

$$\frac{b+a}{b-a} + \frac{b+c}{b-c}$$

$$= \frac{3c+a}{c-a} + \frac{3a+c}{a-c}$$

$$= \frac{3c+a}{c-a} - \frac{3a+c}{c-a} = 2$$

Example 3

If $a^x = b^y = c^z$ and a, b, c are in G.P. prove that x, y, z are in H.P.

Solution:

Given that $a^x = b^y = c^z = k$ (say)

$\therefore$ $a = k^{\frac{1}{x}}$, $b = k^{\frac{1}{y}}$, $c = k^{\frac{1}{z}}$ ---------- (1)

Also given that a, b, c are in G.P.

$\therefore$ $b^2 = ac$ ----------- (2)

Using (1) in (2)

$(k^{\frac{1}{y}})^2 = (k^{\frac{1}{x}})(k^{\frac{1}{z}})$

i.e. $k^{\frac{2}{y}} = k^{\frac{1}{x}+\frac{1}{z}}$

i.e. $\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$

i.e. $\frac{2}{y} = \frac{z+x}{xz}$

i.e. $\frac{y}{2} = \frac{xz}{x+z}$

i.e. $y = \frac{2xz}{x+z}$

$\therefore$ x, y, z are in H.P.

EXERCISE 3.1

1) Find the 4th and 7th terms of the H.P. $\frac{1}{2}$, $\frac{4}{13}$, $\frac{2}{9}$,...

2) The 9th term of an H.P. is $\frac{1}{465}$ and the 20th term is $\frac{1}{388}$. Find the 40th term of the H.P.

3) Prove that $\log_3^2$, $\log_6^2$ and $\log_{12}^2$ are in H.P.

4) If a, b, c are in G.P., prove that $\log_a^m$, $\log_b^m$ and $\log_c^m$ are in H.P.

5) If $\frac{1}{2}$ (x+y), y, $\frac{1}{2}$ (y+z) are in H.P., prove that x, y, z are in G.P.

6) The quantities x, y, z are in A.P. as well as in H.P. Prove that they are also in G.P.

7) If 3 numbers a , b, c are in H.P. show that $\frac{a}{c} = \frac{a-b}{b-c}$

8) If the p^{th} term of an H.P. is q and the q^{th} term is p, prove that its $(pq)^{th}$ term is unity.

9) If a, b, c are in A.P., b, c, a are in G.P. then show that c, a, b are in H.P.

3.2 MEANS OF TWO POSITIVE REAL NUMBERS

Arithmetic Mean of two positive real numbers a and b is defined as

$$\text{A.M.} = \frac{a+b}{2}$$

Geometric Mean of two positive real numbers a and b is defined as

$$\text{G.M.} = +\sqrt{ab}$$

Harmonic Mean of two positive real numbers a and b is defined as

$$\text{H.M.} = \frac{2ab}{a+b}$$

Example 4

Find a) the A.M. of 15 and 25 b) the G.M. of 9 and 4
c) the H.M. of 5 and 45

Solution:

a) $\text{A.M.} = \frac{a+b}{2} = \frac{15+25}{2} = \frac{40}{2} = 20$

b) $\text{G.M.} = +\sqrt{ab} = +\sqrt{9x4} = 6$

c) $\text{H.M.} = \frac{2ab}{a+b} = \frac{2x5x45}{5+45} = \frac{450}{50} = 9$

Example 5

Insert four Arithmetic Means between 5 and 6

Solution:

Let 5, x_1, x_2, x_3, x_4, 6 be in A.P.

\ $t_6 = 6$

i.e. $5 + 5d = 6$

$\therefore \quad d = \frac{1}{5}$

Hence $x_1 = 5 + \frac{1}{5} = \frac{26}{5}$

$x_2 = \frac{26}{5} + \frac{1}{5} = \frac{27}{5}$

$x_3 = \frac{27}{5} + \frac{1}{5} = \frac{28}{5}$

and $x_4 = \frac{28}{5} + \frac{1}{5} = \frac{29}{5}$

The required Arithmetic Means are $\frac{26}{5}, \frac{27}{5}, \frac{28}{5}, \frac{29}{5}$

Example 6

Insert three Geometric Means between $\frac{4}{3}$ and $\frac{3}{4}$

Solution:

Let $\frac{4}{3}$, $x_1, x_2, x_3, \frac{3}{4}$ be in G.P.

$\therefore t_5 = \frac{3}{4}$

i.e. $\frac{4}{3} r^4 = \frac{3}{4}$

$\therefore r = \frac{\sqrt{3}}{2}$

Hence $x_1 = \frac{4}{3} \times \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}}$

$x_2 = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = 1$

and $x_3 = 1 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

The required Geometric Means are $\frac{2}{\sqrt{3}}$, 1, $\frac{\sqrt{3}}{2}$

Example 7

Insert four Harmonic Means between $\frac{1}{9}$ and $\frac{1}{10}$

Solution:

Let $\frac{1}{9}$ $x_1, x_2, x_3, x_4, \frac{1}{10}$ be in H.P.

$\therefore$ 9, $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}$, 10 are in A.P.

$t_6 = 10$

i.e. $9 + 5d = 10 \quad \therefore \quad d = \frac{1}{5}$

Hence $\frac{1}{x_1} = 9 + \frac{1}{5} = \frac{46}{5}$

$\frac{1}{x_2} = \frac{46}{5} + \frac{1}{5} = \frac{47}{5}$

$\frac{1}{x_3} = \frac{47}{5} + \frac{1}{5} = \frac{48}{5}$

and $\frac{1}{x_4} = \frac{48}{5} + \frac{1}{5} = \frac{49}{5}$

The required Harmonic Means are $\frac{5}{46}$, $\frac{5}{47}$, $\frac{5}{48}$, $\frac{5}{49}$,

EXERCISE 3.2

1) Insert 3 Arithmetic Means between 5 and 29.

2) Insert 5 Geometric Means between 5 and 3645.

3) Insert 4 Harmonic Means between $\frac{1}{5}$ and $\frac{1}{20}$

4) The Arithmetic Mean of two numbers is 34 and their Geometric Mean is 16. Find the two numbers.

5) Show that the Arithmetic Mean of the roots of $x^2 - 2ax + b^2 = 0$ is the Geometric Mean of the roots of $x^2 - 2bx + a^2 = 0$ and vice versa.

3.3 RELATION BETWEEN A.M. G.M. AND H.M.

For any two positive unequal real numbers,

i) A.M > G.M > H.M ii) G.M. = $\sqrt{(A.M.) \times (H.M.)}$

Proof :

Denoting the A.M., G.M., and H.M. between two positive unequal real numbers ‘a’ and ‘b’ by A, G, H respectively,

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}$$

Now,

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{a+b-2\sqrt{a}\sqrt{b}}{2} = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} > 0$$

$\therefore$ A > G ---------- (1)

Also

$$G - H = \sqrt{ab} - \frac{2ab}{a+b} = \frac{\sqrt{ab}(a+b)-2ab}{a+b}$$

$$= \frac{\sqrt{ab}\,(a+b)-2\sqrt{ab}\sqrt{ab}}{a+b}$$

$$= \frac{\sqrt{ab}\,\left(a+b-2\sqrt{ab}\right)}{a+b}$$

$$= \frac{\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)^2}{a+b} > 0$$

$\therefore$ G > H ---------- (2)

Combining (1) and (2)

A > G > H

Further

$$A.H. = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right)$$

$$= ab$$

$$= \left(\sqrt{ab}\right)^2$$

$$= G^2$$

$$\therefore G = \sqrt{(A)\,(H)}$$

Hence the proof

Observation:

(i) A.M., G.M., H.M. form a decreasing G.P.

(ii) If we consider the A.M., G.M. and H.M. of two equal positive real numbers each equal to ‘a’ then A.M. = G.M. = H.M. = a.

Example 8

Verify that the A.M., G.M. and H.M. between 25 and 4 form a decreasing G.P.

Solution :

$$A = \frac{a+b}{2} = \frac{25+4}{2} = \frac{29}{2}$$

$$G = \sqrt{ab} = \sqrt{25x4} = 10$$

$$H = \frac{2ab}{a+b} = \frac{2x25x4}{25+4} = \frac{200}{29}$$

Now

$$A - G = \frac{29}{2} - 10 = \frac{29\text{-}10}{2} = \frac{9}{2} > 0$$

$\therefore$ A > G ---------- (1)

Also

$$G - H = 10 - \frac{200}{29} = \frac{290-200}{29} = \frac{90}{29} > 0$$

$\therefore$ G > H ---------------- (2)

Combining (1) and (2)

A > G > H

Further

$$AH = \left(\frac{29}{2}\right)\left(\frac{200}{29}\right)$$

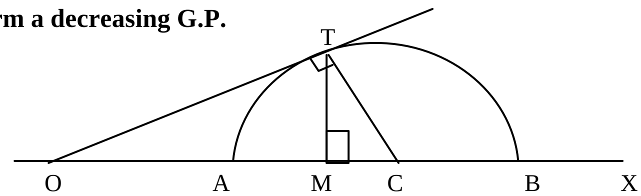
$= 100 = (10)^2$

$= G^2$.

Hence it is verified that A, G, H form a decreasing GP.

Example 9

Represent the A.M, G.M. and H.M. geometrically and hence show that they form a decreasing G.P.



Solution:

From a line OX, cut off OA = a units, OB = b units.

Draw a semicircle on AB as diameter.

Draw OT the tangent to the circle, TM $\perp$ AB.

Let C be the centre of the semi circle.

Now,

$$\frac{a+b}{2} = \frac{OA+OB}{2} = \frac{OC-AC+OC+CB}{2} = \frac{2OC}{2} = OC \ (\because \text{ AC, CB radii})$$

$\therefore$ OC is the A.M. between a and b.

Now

$OT^2 = OA.OB = ab$ ($\because$ OT is tangent and OAB is secant)

i.e. $OT = \sqrt{ab}$

$\therefore$ OT is the G.M. between a and b.

Now

$OT^2 = OM.OC$ ($\because$ $\Delta OTC \,|||\, \Delta OMT$)

i.e. $OM = \frac{OT^2}{OC} = \frac{ab}{\frac{a+b}{2}} = \frac{2ab}{a+b}$

$\therefore$ OM is the H.M. between a and b.

From the right angled Δ OTC,

OC > OT

i.e. A > G ------------- (1)

From the right angled Δ OTM,

OT > OM

i.e. G > H ------------- (2)

combining (1) and (2) we get

A > G > H -------------(3)

Further

$OT^2 = OM.OC$ $\therefore$ OC, OT and OM form a G.P.

i.e. A, G, H form a G.P. -------------(4)

combining (3) and (4) we get that

A.M., G.M., H.M. form a decreasing G.P.

Example 10

If x, y, z be unequal positive real numbers prove that $(x+y)(y+z)(z+x) > 8xyz$

Solution:

Consider x, y

We have A.M. > G.M.

$\therefore \frac{x+y}{2} > \sqrt{xy}$ i.e. $(x+y) > 2\sqrt{xy}$ ------------(1)

Similarly $(y+z) > 2\sqrt{yz}$ --------------(2)

and $(z+x) > 2\sqrt{zx}$ -------------(3)

Multiplying (1), (2) & (3) vertically,

$(x+y)(y+z)(z+x) > \left[2\sqrt{xy}\right]\left[2\sqrt{yz}\right]\left[2\sqrt{zx}\right]$

i.e. $(x+y)(y+z)(z+x) > 8xyz$

EXERCISE 3.3

1) Verify the inequality of the means for the numbers 25 and 36.

2) If a, b, c are three positive unequal numbers in H.P. then show that $a^2 + c^2 > 2b^2$.

3) If x is positive and different from 1 then show that $x + \frac{1}{x} > 2$

3.4 GENERAL CONCEPT OF SEQUENCES

A sequence can be defined (or specified) by

(i) a rule (ii) a recursive relation.

3.4.1 Defining a sequence by a rule

A sequence can be defined by a rule given by a formula for t_n which indicates how to find t_n for a given n.

Example 11

Write out the first four terms of each of the following sequences.

a) $t_n = 3n - 2$ **b)** $t_n = \frac{n^2+1}{n}$ **c)** $t_n = \frac{2n+1}{2n-1}$

d) $t_n = \frac{2^n}{n^2}$ **e)** $< \frac{1+(-1)^n}{2} >$ **f)** $< \frac{n+1}{n-1} >$, $n > 1$

Solution :

a) 1, 4, 7, 10 b) 2, $\frac{5}{2}$, $\frac{10}{3}$, $\frac{17}{4}$ c) 3, $\frac{5}{3}$, $\frac{7}{5}$, $\frac{9}{7}$

d) 2, 1, $\frac{8}{9}$, 1 e) 0, 1, 0, 1 f) 3, 2, $\frac{5}{3}$, $\frac{3}{2}$

Example 12

Determine the range of each of the following sequences

a) $< 2n >$ **b)** $< 2n - 1 >$ **c)** $< 1 + (-1)^n >$

d) $< (-1)^n >$ **e)** $< (-1)^{n-1} >$

Solution:

a) The set of all positive even integers {2, 4, 6, ... }

b) The set of all positive odd integers {1, 3, 5, ... }

c) {0, 2}

d) {-1, 1}

e) {-1, 1}

Example 13

What can you say about the range of the squence
$t_n = n^2 - n + 41$, $n \leq 40$**?**

Solution:

The range is
{41, 43, 47, 53, 61 ... 1601}
This is the set of all prime numbers from 41 to 1601

Example 14

Find an expression for the n^{th} term of each of the following sequences

a) 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, ... **b)** $\frac{3}{2}$, $\frac{5}{4}$, $\frac{7}{6}$, $\frac{9}{8}$, ...

c) **3, 15, 35, 63, ...** **d) 5, 17, 37, 65, ...**

e) $\frac{1}{2}$, - $\frac{2}{3}$, $\frac{3}{4}$, - $\frac{4}{5}$, ... **f) -** $\frac{1}{2}$, $\frac{1}{6}$, - $\frac{1}{12}$, $\frac{1}{20}$, ...

Solution:

a) $t_n = \frac{1}{n^2}$ b) $t_n = \frac{2n+1}{2n}$

c) $t_n = 4n^2-1$ d) $t_n = 4n^2 + 1$

e) $t_n = (-1)^{n+1}\frac{n}{n+1}$ f) $t_n = \frac{(-1)^n}{n^2+n}$

3.4.2 Defining a sequence by a recursive relation.

A recursive relation is a rule given by a formula which enables us to calculate any term of the sequence using the previous terms and the given initial terms of the sequence.

Example 15

Find the first seven terms of the sequence given by the recursive relation,

$\mathbf{a_1 = 1, \quad a_2 = 0, \quad a_n = 2a_{n-1} - a_{n-2}, \ n>2}$

Solution:

$a_3 = 2a_2-a_1 = 0-1 = -1$
$a_4 = 2a_3-a_2 = -2-0 = -2$
$a_5 = 2a_4-a_3 = -4+1 = -3$
$a_6 = 2a_5-a_4 = -6+2 = -4$
$a_7 = 2a_6-a_5 = -8+3 = -5$

The first seven terms are 1, 0, -1, -2, -3, -4, -5

Example 16

Find the first 10 terms of the sequence

$\mathbf{a_1 = 1, \ a_2 = 1, \ a_{n+1} = a_n + a_{n-1}, \ n > 2}$

Solution:

$a_3 = a_2+a_1 = 1+1 = 2$

$a_4 = a_3+a_2 = 2+1 = 3$

$a_5 = a_4+a_3 = 3+2 = 5$
$a_6 = a_5+a_4 = 5+3 = 8$
$a_7 = a_6+a_5 = 8+5 = 13$
$a_8 = a_7+a_6 = 13+8 = 21$
$a_9 = a_8+a_7 = 21+13 = 34$
$a_{10} = a_9+a_8 = 34+21 = 55$

The first ten terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

Observation :

This type of sequence is called Fibonacci sequence.

Example 17

Show that : (i) $t_n = 2^{n+1} - 3$ and (ii) $a_1 = 1$, $a_n = 2a_{n-1} + 3, n \geq 2$ represent the same sequence.

Solution:

(i) $t_n = 2^{n+1} - 3$
$t_1 = 2^2-3 = 1$
$t_2 = 2^3-3 = 5$
$t_3 = 2^4-3 = 13$
$t_4 = 2^5-3 = 29$
$t_5 = 2^6-3 = 61$ and so on.
The sequence is 1, 5, 13, 29, 61...

(ii) $a_1 = 1$
$a_n = 2a_{n-1}+3, \quad n \geq 2$
$a_2 = 2a_1+3 = 2+3 = 5$
$a_3 = 2a_2+3 = 10+3 = 13$
$a_4 = 2a_3+3 = 26+3 = 29$
$a_5 = 2a_4+3 = 58+3 = 61$ and so on.

The sequence is 1, 5, 13, 29, 61, ...
i.e. The two sequences are the same.

Observation

There may be sequences which defy an algebraic representation. For example, the sequence of prime numbers 2, 3, 5, 7, 11, 13, ... Mathematicians are still striving hard to represent all prime numbers by a single algebraic formula. Their attempts have not been successful so far.

EXERCISE 3.4

1) Write out the first 5 terms of each of the following sequences

(a) $< \frac{n+1}{n!} >$ (b) $< \frac{(-1)^{n-1}}{n+1} >$ (c) $< \frac{1}{n^n} >$ (d) $< \frac{1-(-1)^n}{n+1} >$

(e) $< n\, 2^{2n-1} >$ (f) $<(-1)^n>$ (g) $<6n-1>$

2) Write out the first 7 terms of the sequence

$$t_n = \begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd} \\ 3\left(\frac{n}{2}+1\right), & \text{if } n \text{ is even} \end{cases}$$

3) Find the range of each of the following sequences

(a) $< 1+(-1)^{n+1} >$ (b) $< (-1)^{n+1} >$

4) Find the general term of each of the following sequences

(a) 1, 4, 9, 16, 25 ...

(b) 3, 7, 11, 15, 19, 23, ...

(c) 2.1, 2.01, 2.001, 2.0001, ...

(d) 0, 3, 8, 15, ...

(e) $\frac{10}{3}$, $\frac{20}{9}$, $\frac{30}{27}$, $\frac{40}{81}$, ...

5) Find the first 6 terms of the sequence specified by the recursive relation

(a) $a_1 = 1,\ a_n = \frac{a_{n-1}}{2}$, $n > 1$ (b) $a_1 = 5,\ a_n = -2a_{n-1},\ n > 1$

(c) $a_1 = 1,\ a_n = 3a_{n-1} + 1,\ n > 1$ (d) $a_1 = 2,\ a_n = 2a_{n-1} + n,\ n > 1$

(e) $a_1 = 1,\ a_n = a_{n-1} + n^2,\ n > 1$ (f) $a_1 = 2,\ a_2 = 1,\ a_n = a_{n-1} - 1,\ n > 2$

(g) $a_1 = 1,\ a_2 = 1\ a_n = (a_{n-1})^2 + 2,\ n > 2$ (h) $a_1 = 1, a_2 = -1, a_n = a_{n-2} + 2,\ n > 2$

3.5 COMPOUND INTEREST

In compound interest, the interest for each period is added to the principal before the interest is calculated for the next period. Thus the interest earned gets reinvested and in turn earns interest.

The formula to find the amount under compound interest is given by

$$\boxed{A = P(1+i)^n}, \quad \text{where } i = \frac{r}{100}$$

Here P = Principal

A = Amount

r = Rate of Interest

i = Interest on unit sum for one year

Also the present value P is given by $\boxed{P = \frac{A}{(1+i)^n}}$

Observation :

(i) The amounts under compound interest form a G.P.

(ii) If the interest is paid more than once in a year the rate of interest is what is called nominal rate.

(iii) If the interest is paid k times a year then *i* must be replaced by $\frac{i}{k}$ and n by nk.

(iv) If a certain sum becomes N times in T years then it will become N^n times in T x n years.

Example 18

Find the compound interest on Rs. 1,000 for 10 years at 5% per annum.

Solution:

A = P $(1+i)^n$

= 1000 $(1+0.05)^{10}$

= 1000 $(1.05)^{10}$

= Rs. 1629

Compound Interest = A - P

= 1629 - 1000

= Rs. 629.

Logarithmic calculation

log 1.05 = 0.0212

10 x

0.2120

log 1000 = 3.0000 +

3.2120

Antilog 3.2120

= 1629

Example 19

Find the compound interest on Rs. 1,000 for 10 years at 4% p.a., the interest being paid quarterly.

Solution:

A = $P(1+i)^n$

= $1000\ (1+0.01)^{40}$

= $1000\ (1.01)^{40}$

= Rs. 1486

Compound interest = A - P

= 1486 - 1000

= Rs. 486.

Logarithmic calculation

log 1.01 = 0.0043
40 x

0.1720
log 1000 = 3.0000 +

3.1720

Antilog 3.1720
= 1486

Example 20

A person deposits a sum of Rs. 10,000 in the name of his new-born child. The rate of interest is 12% p.a. What is the amount that will accrue on the 20th birthday of the beneficiary if the interest is compounded monthly.

Solution:

A = $P(1+i)^n$

= $10000\ (1+0.01)^{240}$

= $10000\ (1.01)^{240}$

= Rs. 1,07,600

Logarithmic calculation

log 1.01 = 0.0043
240 x

1.0320
log 10000 = 4.0000 +

5.0320

Antilog 5.0320
= 1,07,600

Example 21

The population of a city in 1987 was 50,000. The population increases at the rate of 5% each year. Find the population of the city in 1997.

Solution:

A = $P(1+i)^n$

= $50000\ (1+0.05)^{10}$

= $50000\ (1.05)^{10}$

= 81,470

Logarithmic calculation

log 1.05 = 0.0212
10 x

0.2120
log 50000 = 4.6990 +

4.9110

Antilog 4.9110
= 81,470

Example 22

A machine depreciates in value each year at the rate of 10% of its value at the begining of a year. The machine was purchased for Rs. 10,000. Obtain its value at the end of the 10th year.

Solution:

$$A = P(1-i)^n$$
$$= 10000\,(1-0.1)^{10}$$
$$= 10000\,(0.9)^{10}$$
$$= \text{Rs. } 3,483$$

Logarithmic calculation

$\log 0.9 = \bar{1}.9542$

10 x

$\bar{1}.5420$ +

log 10000 = 4.0000

3.5420

Antilog 3.5420 = 3,483

Example 23

Find the present value of an amount of Rs. 12,000 at the end of 5 years at 5% C.I.

Solution:

$$P = \frac{A}{(1+i)^n}$$
$$= \frac{12000}{(1+0.05)^5}$$
$$= \frac{12000}{(1.05)^5}$$
$$= \text{Rs. } 9,401$$

Logarithmic calculation

log 1.05 = 0.0212

5 x

0.1060

log 12000 = 4.0792

0.1060 -

3.9732

Antilog 3.9732 = 9,401

Example 24

What sum will amount to Rs. 5,525 at 10% p.a. compounded yearly for 13 years.

Solution:

$$P = \frac{A}{(1+i)^n}$$

$$= \frac{5525}{(1+0.1)^{13}}$$

$$= \frac{5525}{(1.1)^{13}}$$

= Rs. 1,600

Logarithmic calculation

log 1.1 = 0.0414
13 x
0.5382
log 5525 = 3.7423
0.5382 -
3.2041
Antilog 3.2041
= 1,600

Example 25

At what rate percent p.a. C.I. will Rs. 2,000 amount to Rs. 3,000 in 3 years if the interest is reckoned half yearly.

Solution:

A = P $(1+i)^n$

$3000 = 2000 \left(1+\frac{i}{2}\right)^{3\times2}$

$= 2000 \left(1+\frac{i}{2}\right)^{6}$

=> $\left(1+\frac{i}{2}\right)^{6} = \frac{3000}{2000}$

=> $\left(1+\frac{i}{2}\right) = (1.5)^{\frac{1}{6}} = 1.07$

=> $\frac{i}{2} = 0.07$

i.e. $\frac{r}{100} = 0.14$

∴ r = 14%

Logarithmic calculation

log 1.5 = 0.1761
÷ 6
0.02935
Antilog 0.02935
= 1.07

Example 26

How long will it take for a given sum of money to triple itself at 13% C.I.?

Solution:

A = $P(1+i)^n$

3P = $P(1+0.13)^n$

i.e. 3 = $(1.13)^n$

Taking logarithm,

$$\log 3 = n \log 1.13$$

i.e. $n = \frac{\log 3}{\log 1.13} = \frac{0.4771}{0.0531}$

$= 8.984 = 9$ years (nearly)

Logarithmic calculation

$\log 0.4771 = \bar{1}.6786$

$\log 0.0531 = \bar{2}.7251$ -

0.9535

Antilog 0.9535

= 8.984

3.5.1 Effective rate of interest:

When interest is compounded more than once in a year the rate of interest is called nominal rate.

The interest rate, which compounded once in a year gives the same interest as the nominal rate is called effective rate.

Obvisously Effective rate > nominal rate.

Let i be the nominal interest per unit sum per year compounded k times a year and j the corresponding effective interest on unit sum per year. Then for the principal P,

$$P(1+j) = P\left(1+\frac{i}{k}\right)^k$$

i.e.

$$\boxed{j = \left(1+\frac{i}{k}\right)^k - 1}$$

Example 27

Find the effective rate of interest when the rate of interest is 15% and the interest is paid half yearly.

Solution:

$j = \left(1+\frac{i}{k}\right)^k - 1$

$= \left(1+\frac{0.15}{2}\right)^2 - 1$

$= (1 + 0.075)^2 - 1$

$= (1.075)^2 - 1 = 1.155 - 1$

$= 0.155 = 15.5\%$

Logarithmic calculation

$\log 1.075 = 0.0314$

2

0.0628

Antilog 0.0628

= 1.155

Example 28

Find the effective rate of interest for the interest rate 16% if interest is compounded once in two months.

Solution:

$$j = (1 + \frac{i}{k})^k - 1$$
$$= (1+\frac{0.16}{6})^6 - 1$$
$$= (1+0.027)^6 - 1$$
$$= (1.027)^6 - 1$$
$$= 1.174 - 1$$
$$= 0.174$$
$$= 17.4\%$$

Logarithmic calculation

log 1.027 = 0.0116
$\times$ 6
0.0696

Antilog 0.0696
= 1.174

Example 29

A finance company offers 16% interest compounded annually. A debenture offers 15% interest compounded monthly. Advise which is better.

Solution:

Convert the nominal rate 15% to effective rate.

$$j = (1+\frac{i}{k})^k - 1$$
$$= (1+\frac{0.15}{12})^{12} - 1$$
$$= (1+0.0125)^{12} - 1$$
$$= (1.0125)^{12} - 1$$
$$= 1.164 - 1$$
$$= 0.164$$
$$= 16.4\ \%$$

Logarithmic calculation

log 1.0125 = 0.0055
12 x
0.0660

Antilog 0.0660
= 1.164

Comparing, we conclude that 15% compounded monthly is better.

EXERCISE 3.5

1) How much will Rs. 5,000 amount to at 12% p.a. C.I. over 15 years?

2) Find the C.I. for Rs. 4,800 for 3 years at 4% p.a. when the interest is paid
i) annually ii) half yearly

3) A person invests Rs. 2,000 at 15%. If the interest is compounded monthly, what is the amount payable at the end of 25 years?

4) A machine depreciates in value each year at the rate of 10% of its value at the begining of a year. The machine was purchased for Rs. 20,000. Obtain the value of the machine at the end of the fourth year.

5) Find the present value of Rs. 2,000 due in 4 years at 4% C.I.

6) Mrs. Kalpana receives Rs. 4888 as compound interest by depositing a certain sum in a 10% fixed deposit for 5 years. Determine the sum deposited by her.

7) At what rate percent per annum C.I. will Rs. 5000 amount to Rs. 9035 in 5 years, if C.I. is reckoned quarterly?

8) In how many years will a sum of money treble itself at 5% C.I. payable annually?

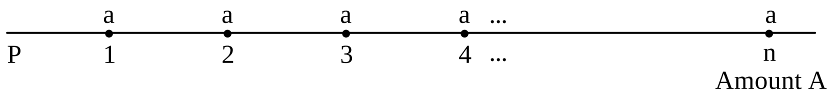
9) Find the effective rate of interest when the interest is 15% paid quarterly

10) Find the effective rate corresponding to the nominal rate of 12% compounded half yearly.

3.6 ANNUITIES

A sequence of equal payments at equal intervals of time is called an annuity. If the payments are made at the end of each period the annuity is called *immediate annuity* or *ordinary annuity.* If the payments are made at the begining of each period the annuity is called *annuity due.* Annuity generally means ordinary annuity.

3.6.1 Immediate Annuity



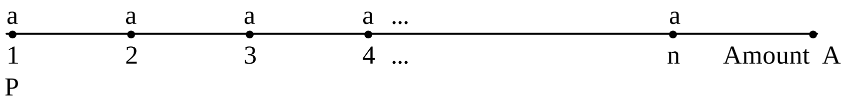
If equal payments ‘a’ are made at the end of each year for n years, then the Amount

$$A = \frac{a}{i}\left[(1+i)^n - 1\right]$$

Also if P is the present value then

$$P = \frac{a}{i}\left[1 - (1+i)^{-n}\right]$$

3.6.2 Annuity Due



If equal payments ‘a’ are made at the beginning of each year for n years, then the Amount

$$A = \frac{a}{i}(1+i)\left[(1+i)^n - 1\right]$$

Also if P is the present value, then

$$P = \frac{a}{i}(1+i)\left[1 - (1+i)^{-n}\right]$$

Example 30

Find the amount of annuity of Rs. 2,000 payable at the end of each year for 4 years if money is worth 10% compounded annually.

Solution:

$$A = \frac{a}{i}\left[(1+i)^n - 1\right]$$

$$= \frac{2000}{0.1}\left[(1.1)^4 - 1\right]$$

$= \frac{2000}{\frac{1}{10}}$ [1.464-1]

= 20000 [0.464]

= Rs. 9,280

Logarithmic calculation

log 1.1 = 0.0414
4 x
0.1656
Antilog 0.1656
= 1.464

Example 31

Find the amount of an ordinary annuity of 12 monthly payments of Rs. 1,000 that earn interest at 12% per year compounded monthly.

Solution:

$A = \frac{a}{i} [(1+i)^n - 1]$

$= \frac{1000}{0.01} [(1.01)^{12} - 1]$

$= \frac{2000}{\frac{1}{100}}$ [1.127-1]

= 100000 [0.127]

= Rs. 12,700

Logarithmic calculation

log 1.01 = 0.0043
12 x
0.0516
Antilog 0.0516
= 1.127

Example 32

A bank pays 8% interest compounded quarterly. Determine the equal deposits to be made at the end of each quarter for 3 years so as to receive Rs. 3,000 at the end of 3 years.

Solution:

$A = \frac{a}{i} [(1+i)^n - 1]$

i.e. $3000 = \frac{a}{0.02} [(1.02)^{12} - 1]$

=> 60 = a [1.2690-1]

=> 60 = a [0.2690]

$\therefore$ a $= \frac{60}{0.2690}$

= Rs. 223

Logarithmic calculation

log 1.02 = 0.0086
12 x
0.1032
Antilog 0.1032
= 1.2690

log 60 = 1.7782
log 0.2690 = $\bar{1}$.4298
2.3484 -
Antilog 2.3484
= 223.0

Example 33

What is the present value of an annuity of Rs. 750 p.a. received at the end of each year for 5 years when the discount rate is 15%.

Solution :

$$P = \frac{a}{i}[1-(1+i)^{-n}]$$

$$= \frac{750}{0.15}[1-(1.15)^{-5}]$$

$$= \frac{75000}{15}[1-0.4972]$$

$$= 5000\,[0.5028]$$

= Rs. 2514

Logarithmic calculation

log 1.15 = 0.0607

- 5 x

- 0.3035

$= \bar{1}.6965$

Antilog $\bar{1}.6965$

= 0.4972

Example 34

An equipment is purchased on an instalment basis such that Rs. 5000 is to be paid on the signing of the contract and four yearly instalments of Rs. 3,000 each payable at the end of first, second, third and fourth year. If the interest is charged at 5% p.a., find the cash down price.

Solution:

$$P = \frac{a}{i}[1-(1+i)^{-n}]$$

$$= \frac{3000}{0.05}[1-(1.05)^{-4}]$$

$$= \frac{3000}{\frac{5}{100}}[1-0.8226]$$

$$= \frac{300000}{5}[0.1774]$$

$$= 60000\,[0.1774]$$

= Rs. 10644

Logarithmic calculation

log 1.05 = 0.0212

-4 x

-0.0848

$= \bar{1}.9152$

Antilog $\bar{1}.9152$

= 0.8226

Cash down payment = Rs. 5,000

∴ Cash down price = Rs. (5000 + 10644) = Rs. 15,644

Example 35

A person borrows Rs. 5000 at 8% p.a. interest compounded half yearly and agrees to pay both the principal and interest at 10 equal instalments at the end of each of six months. Find the amount of these instalments.

Solution:

$P = \frac{a}{i}[1-(1+i)^{-n})$

$5000 = \frac{a}{0.04}[1-(1.04)^{-10}]$

$= \frac{a}{0.04}[1-0.6761]$

$= \frac{a}{0.04}[0.3239]$

i.e. $200 = a\,[0.3230]$

$a = \frac{200}{0.3239}$

= Rs. 617.50

Logarithmic calculation

log 1.04 = 0.0170
-10 x
-0.1700
= $\bar{1}.8300$
Antilog $\bar{1}.8300$
= 0.6761

log 200 = 2.3010
log 0.3239 = $\bar{1}.5104$ -
2.7906
Antilog 2.7906
= 617.50

Example 36

Machine X costs Rs. 15,000 and machine Y costs Rs. 20,000. The annual income from X and Y are Rs. 4,000 and Rs. 7,000 respectively. Machine X has a life of 4 years and Y has a life of 7 years. Find which machine may be purcahsed. (Assume discount rate 8% p.a.)

Solution:

<u>**Machine X**</u>

Present value of outflow = Rs. 15,000

Present value of inflows

$= \frac{a}{i}[1-(1+i)^{-n}]$

$= \frac{4000}{0.08}[1-(1.08)^{-4}]$

Logarithmic calculation

log 1.08 = 0.0334
-4 x
-0.1336
= $\bar{1}.8664$
Antilog $\bar{1}.8664$
= 0.7352

$= \frac{400000}{8}$ [1-0.7352]

= 50000 [0.2648]

Rs. 13,240

Present inflow is less than present outflow.

∴ Net outflow = Rs. (15,000-13,240)

= Rs. 1760

Machine Y

Present value of outflow = Rs. 20,000

Present value of inflows

$= \frac{a}{i}[1-(1+i)^{-n}]$

$= \frac{7000}{0.08}[1-(1.08)^{-7}]$

$= \frac{7000}{\frac{8}{100}}[1-0.5837]$

$= \frac{700000}{8}[0.4163]$

= 87500 [0.4163]

= Rs. 36,420

Logarithmic calculation

log 1.08 = 0.0334

-7 x

-0.2338

$= \bar{1}.7662$

Antilog $\bar{1}.7662$

= 0.5837

log 87500 = 4.9420

log 0.4163 = $\bar{1}.6194$ +

4.5614

Antilog = 4.5614

= 36,420

Present inflow is more than present outflow

∴ Net inflow = Rs. (36,420-20000)

= Rs. 16,420

∴ Machine Y may be purchased

Example 37

If I deposit Rs. 500 every year for a period of 10 years in a bank which gives C.I. of 5% per year, find out the amount I will receive at the end of 10 years.

Solution:

$$A = \frac{a}{i}(1+i)[(1+i)^n-1]$$

$$= \frac{500}{0.05}(1.05)[(1.05)^{10}-1]$$

$$= \frac{525}{0.05}[1.629-1]$$

$$= \frac{525}{\frac{5}{100}}[0.629]$$

$$= \frac{52500}{5}[0.629]$$

$$= 10500\,[0.629]$$

= Rs. 6604.50

Logarithmic calculation

log 1.05 = 0.0212
10 x
0.2120
Antilog 0.2120
= 1.629

Example 38

A sum of Rs. 1,000 is deposited at the beginning of each quarter in a S.B. account that pays C.I. 8% compounded quarterly. Find the amount in the account at the end of 3 years.

Solution :

$$A = \frac{a}{i}(1+i)[(1+i)^n-1]$$

$$= \frac{1000}{0.02}(1.02)[(1.02)^{12}-1]$$

$$= \frac{1020}{0.02}[1.269-1]$$

$$= \frac{1020}{\frac{2}{100}}[0.269]$$

$$= \frac{102000}{2}[0.269]$$

$$= 51000\,[0.269]$$

= Rs. 13,719

Logarithmic calculation

log 1.02 = 0.0086
12 x
0.1032
Antilog 0.1032
= 1.269

Example 39

What equal payments made at the beginning of each month for 3 years will accumulate to Rs. 4,00,000, if money is worth 15% compounded monthly.

Solution :

$$A = \frac{a}{i}(1+i)[(1+i)^n-1]$$

$$400{,}000 = \frac{a}{0.0125}(1.0125)[(1.0125)^{36}-1]$$

ie. $5000 = a(1.0125)[1.578-1]$

$= a(1.0125)(0.578)$

$$\therefore \quad a = \frac{5000}{(1.0125)(0.578)}$$

= Rs. 8,543

Logarithmic calculation

log 1.01125 = 0.0055

36 x

0.1980

Antilog 0.1980

= 1.578

log 1.0125 = 0.0055

log 0.578 = $\bar{1}.7619$ +

$\bar{1}.7674$

log 5000 = 3.6990

$\bar{1}.7674$ -

3.9316

Antilog 3.9316

= 8,543

Example 40

Find the present value of an annuity due of Rs. 200 p.a. payable annualy for 2 years at 4% p.a.

Solution:

$$P = \frac{a}{i}(1+i)[1-(1+i)^{-n}]$$

$$= \frac{200}{0.04}(1.04)[1-(1.04)^{-2}]$$

$$= \frac{208}{\frac{4}{100}}[1-0.9247]$$

$$= \frac{20800}{4}[0.0753]$$

= 5,200 [0.0753]

= Rs. 391.56

Logarithmic calculation

log 1.04 = 0.0170

-2 x

-0.0340

= $\bar{1}.9660$

Antilog $\bar{1}.9660$

= 0.9247

EXERCISE 3.6

1) Find the future value of an ordinary annuity of Rs. 1000 a year for 5 years at 7% p.a. compounded annually.

2) A man deposits Rs. 75 at the end of each of six months in a bank which pays interest at 8% compounded semi annually. How much is to his credit at the end of ten years?

3) Find the present value of an annuity of Rs. 1200 payable at the end of each of six months for 3 years when the interest is earned at 8% p.a. compounded semi annually.

4) What is the present value of an annuity of Rs. 500 p.a. received for 10 years when the discount rate is 10% p.a.?

5) What is the present value of an annuity that pays Rs. 250 per month at the end of each month for 5 years, assuming money to be worth 6% compounded monthly?

6) Machine A costs Rs. 25,000 and machine B costs Rs. 40,000. The annual income from the machines A and B are Rs.8,000 and Rs. 10,000 respectively. Machine A has a life of 5 years and machine B has a life of 7 years. Which machine may be purchased, discount rate being 10% p.a.?

7) A man wishes to pay back his debts of Rs 3,783 due after 3 years by 3 equal yearly instalments. Find the amount of each instalment, money being worth 5% p.a. compounded annually.

8) A person purchases a house of worth Rs. 98,000 on instalment basis such that Rs. 50,000 is to be paid in cash on the signing of the contract and the balance in 20 equal instalments which are to be paid at the end of each year. Find each instalment to be paid if interest be reckoned 16% p.a. compounded annually.

9) If I deposit Rs. 1,000 every year for a period of 5 years in a bank which gives a C.I. of 5% p.a. find out the amount at the end of 5 years.

10) A sum of Rs. 500 is deposited at the beginning of each year. The rate of interest is 6% p.a. compounded annually. Find the amount at the end of 10 years.

11) A firm purchases a machine under an instalment plan of Rs. 10,000 p.a. for 8 years. Payments are made at the beginning of each year. What is the present value of the total cash flow of the payments for interest at 20%.?

12) A bank pays interest at the rate of 8% p.a. compounded quarterly. Find how much should be deposited in the bank at the beginning of each of 3 months for 5 years in order to accumulate to Rs. 10,000 at the end of 5 years.

13) What equal payments made at the beginning of each year for 10 years will pay for a machine priced at Rs. 60,000, if money is worth 5% p.a. compounded annually?

EXERCISE 3.7

Choose the correct answer

1) The progression formed by the reciprocals of the terms of an H.P. is

(a) A.P. (b) G.P. (c) H.P. (d)none of these

2) $\frac{1}{8}$, x, $\frac{3}{2}$ are in H.P. then x is equal to

(a) $\frac{3}{13}$ (b) $\frac{4}{13}$ (c) $\frac{5}{13}$ (d) $\frac{6}{13}$

3) The Arithmetic Mean between a and b is

(a) $\frac{ab}{2}$ (b) $\frac{a+b}{2}$ (c) $\sqrt{ab}$ (d) $\frac{a-b}{2}$

4) The Geometric Mean between 3 and 27 is

(a) 15 (b) 12 (c) 19 (d)none of these

5) The Harmonic Mean between 10 and 15 is

(a) 12 (b) 25 (c) 150 (d) 12.5

6) The Harmonic Mean between the roots of the equation $x^2 - bx + c = 0$ is

(a) $\frac{2b}{c}$ (b) $\frac{2c}{b}$ (c) $\frac{2bc}{b+c}$ (d)none of these

7) If the Arithmetic Mean and Harmonic Mean of the roots of a qudradratic equation are $\frac{3}{2}$ and $\frac{4}{3}$ respectively then the equation is

[redacted] [redacted]

(c) $x^2 - 3x - 4 = 0$ (d) $x^2 + 2x + 3 = 0$

8) The A.M., G.M. and H.M.between two unequal positive numbers are themselves in

(a) G.P. (b) A.P. (c) H.P. (d)none of these

9) If A, G, H are respectively the A.M., G.M. and H.M between two different positive real numbers then

(a) $A > G > H$ (b) $A < G > H$ (c) $A < G < H$ (d) $A>G<H$

10) If A, G, H are respectively the A.M., G.M and H.M. between two different positive numbers then

(a) $A = G^2H$ (b) $G^2 = AH$ (c) $A^2 = GH$ (d) $A = GH$

11) For two positive real numbers G.M. = 300, H.M. = 180 their A.M. is

(a) 100 (b) 300 (c) 200 (d) 500

12) For two positive real numbers, A.M. = 4, G.M. = 2 then the H.M. between them is

(a) 1 (b) 2 (c) 3 (d) 4

13) The fifth term of the sequence $< \frac{(-1)^{n+1}}{n} >$ is

(a) $\frac{1}{5}$ (b) $-\frac{1}{5}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{4}$

14) In the sequence 1000, 995, 990, ... find n for which t_n is the first negative term.

(a) 201 (b) 204 (c) 202 (d) 203

15) The range of the sequence $<2+(-1)^n>$ is

(a) N (b) R (c) {3, 4} (d) {1, 3}

16) The successive amounts on a principal carrying S.I. for one year, two years, three years form.

(a) an A.P. (b) a G.P. (c) an H.P. (d)none of these

17) The successive amounts on a principal carrying C.I. forms

(a) an A.P. (b) a G.P. (c) an H.P. (d)none of these

18) The compounded interest on Rs. P after T years at R% p.a., compounded annually is

(a) Rs. P $[(1+\frac{R}{100})^T+1]$ (b) Rs. P $[(1+\frac{R}{100})^T-1]$

(c) Rs. P $[(1+\frac{R}{100})^T-100]$ (d) Rs. P $[(1+\frac{R}{100})^T+100]$

19) The compound interest on Rs. 400 for 2 years at 5% p.a. compounded annually is

(a) Rs. 45 (b) Rs. 41 (c) Rs. 20 (d) Rs. 10

20) The interest on Rs. 24,000 at the rate of 5% C.I. for 3 years.

(a) Rs. 3,783 (b) Rs. 3,793 (c) Rs. 4,793 d) Rs. 4,783

21) The difference between S.I. and C.I. on a sum of money at 5% p.a. for 2 years is Rs. 25. Then the sum is.

(a) Rs. 10,000 (b) Rs. 8,000 (c) Rs. 9,000 (d) Rs. 2,000

22 If Rs. 7,500 is borrowed at C.I. at the rate of 4% p.a., then the amount payable after 2 years is

(a) Rs. 8,082 (b) Rs. 7,800 (c) Rs. 8,100 (d) Rs. 8,112

23) Rs. 800 at 5% p.a. C.I. will amount to Rs. 882 in

(a) 1 year (b) 2 years (c) 3 years (d) 4 years

24) A sum amounts to Rs. 1352 in 2 years at 4% C.I. Then the sum is

(a) Rs. 1300 (b) Rs. 1250 (c) Rs. 1260 (d) Rs. 1200

25) The principal which earns Rs. 132 as compound interest for the second year at 10% p.a. is

(a) Rs. 1000 (b) Rs. 1200 (c) Rs. 1320 (d)none of these

26) A sum of Rs. 12,000 deposited at CI becomes double after 5 years. After 20 years it will become

(a) Rs. 1,20,000 (b) Rs. 1,92,000 (c) Rs. 1,24,000 (d) Rs. 96,000

27) A sum of money amounts to Rs. 10,648 in 3 years and Rs. 9,680 in 2 years. The rate of C.I. is

(a) 5% (b)10% (c) 15% (d) 20%

28) The value of a machine depreciates every year at the rate of 10% on its value at the begining of that year. If the present value of the machine is Rs. 729, its worth 3 years ago was

(a) Rs. 947.10 (b) Rs. 800 (c) Rs. 1000 (d) Rs. 750.87

29) At compound interest if a certain sum of money doubles in n years then the amount will be four fold in

(a) $2n^2$ years (b) n^2 years (c) 4n years (d) 2n years

30) A sum of money placed at C.I. doubles in 5 years. It will become 8 times in

(a) 15 years (b) 9 years (c) 16 years (d) 18 years

31) A sum of money at C.I. amounts to thrice itself in 3 years. It will be 9 times in

(a) 9 years (b) 6 years (c) 12 years (d) 15 years

32) If i is the interest per year on a unit sum and the interest is compounded k times a year then the corresponding effective rate of interest on unit sum per year is given by

(a) $(1+\frac{k}{i})^{i}-1$ (b) $(1+\frac{k}{i})^{\frac{i}{k}}-1$ (c) $(1+\frac{i}{k})^{k}-1$ (d)none of these

33) If i is the interest per year on a unit sum and the interest is compounded once in k months in a year then the corresponding effective rate of interest on unit sum per year is given by

(a) $(1+\frac{12}{k}i)^{\frac{k}{12}}-1$ b) $(1+\frac{ki}{12})^{\frac{12}{k}}-1$ (c) $(1+\frac{ki}{12})^{\frac{12}{k}}+1$ (d) none of these

ANALYTICAL GEOMETRY 4

The word "Geometry" is derived from the Greek word "geo" meaning "earth" and "metron" meaning "measuring". The need of measuring land is the origin of geometry.

The branch of mathematics where algebraic methods are employed for solving problem in geometry is known as Analytical Geometry. It is sometimes called cartesian Geometry after the french mathematician Des-Cartes.

4.1 LOCUS

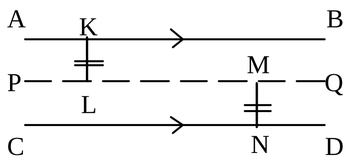
Locus is the path traced by a moving point under some specified geometrical condition The moving point is taken as P(x,y).

Equation of a locus:

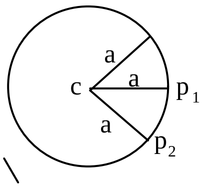
Any relation in x and y which is satisfied by every point on the locus is called the equation of the locus.

For example

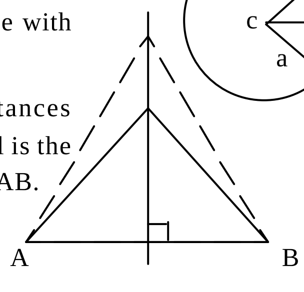
(i) The locus of a point equidistant from two given lines is the line parallel to each of the two lines and midway between them.



(ii) The locus of a point whose distance from a fixed point is constant is a circle with the fixed point as its centre.



(iii) The locus of a point whose distances from two points A and B are equal is the perpendicular bisector of the line AB.



Exampe 1

Find the locus of a point which moves so that its distance from the point (2,5) is always 7 units.

Solution:

Let P(x,y) be the moving point. The given fixed point is A(2,5).

Now, PA = 7

$\therefore$ $PA^2 = 7^2 = 49$

(ie) $(x-2)^2 + (y-5)^2 = 49$

$x^2 - 4x + 4 + y^2 - 10y + 25 - 49 = 0$

$\therefore$ the locus is $x^2 + y^2 - 4x - 10y - 20 = 0$

Example 2

Find the equation of locus of the point which is equidistant from (2,-3) and (4,7)

Solution:

Let P(x,y) be the moving point. Let the given points be A (2, -3)and B(4, 7).

Given that PA = PB $\therefore PA^2 = PB^2$

$(x-2)^2 + (y+3)^2 = (x-4)^2 + (y-7)^2$

i.e., $x + 5y - 13 = 0$

Example 3

A point P moves so that the points P, A(1,-6) and B(2,5) are always collinear. Find the locus of P.

Solution:

Let P(x,y) be the moving point. Given that P,A,B are collinear.

$\therefore$ Area of $\Delta PAB = 0$

ie $\frac{1}{2}$ [x(-6-5) + 1(5-y) + 2(y+6)] = 0

$\therefore$ 11x - y - 17 = 0 is the required locus.

EXERCISE 4.1

1) Find the locus of a point which moves so that it is always equidistant from the two points (2,3) and (-2,0)

2) Find the locus of a point P which moves so that PA = PB where A is (2,3) and B is (4,-5)

3) A point moves so that its distance from the point (-1,0) is always three times its distance from the point (0,2). Find its locus.

4) Find the locus of a point which moves so that its distance from the point (3,7) is always 2 units.

5) A and B are two points (-2,3), (4,-5) Find the equation to the locus point P such that $PA^2 - PB^2 = 20$

6) Find the equation to the locus of a point which moves so that its distance from the point (0,1) is twice its distance from the x axis.

7) Find the perpendicular bisector of the straight line joining the points (2,-3) and (3,-4)

8) The distance of a point from the origin is five times its distance from the y axis. Find the equation of the locus.

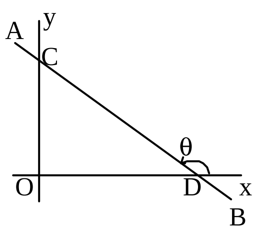
9) Find the locus of the point which moves such that its distances from the points (1,2), (0,-1) are in the ratio 2:1

10) A point P moves so that P and the points (2,3) and (1,5) are always collinear. Find the locus of P.

4.2 EQUATION OF LINES

RECALL

The line AB cuts the axes at D and C respectively. θ is the angle made by the line AB with the positive direction of x - axis.



$\tan\theta$ = slope of the line AB is denoted by m. OD is called the x -intercept OC is called the y - intercept.

Slope Point Form:

Equation of a straight line passing through a given point (x_1,y_1) and having a given slope m is $y-y_1 = m(x-x_1)$

Slope Intercept Form:

The equation of a straight line with slope 'm' and y intercept 'c' is $y = mx+c$.

Two Point Form:

The equation of a straight line passing through the points (x_1,y_1) and (x_2,y_2) is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

When the two points (x_1, y_1) and (x_2, y_2) are given, then the slope of the line joining them is

$$\frac{y_2-y_1}{x_2-x_1}$$

Intercept Form:

Equation of a line with x intercept a and y intercept b is

$$\frac{x}{a} + \frac{y}{b} = 1$$

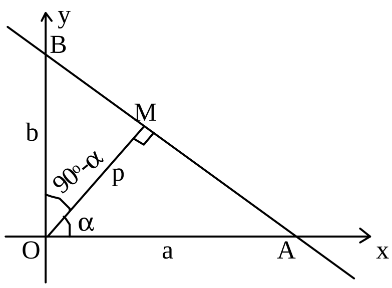
General Form:

Any equation of the first degree in x, y of the form $Ax+By+C = 0$ represents equation of a straight line with slope $-\left(\frac{A}{B}\right)$

4.2.1 Normal Form:

When the length of the ⊥ r from the origin to a straight line is p and the inclination of the ⊥ r with x -axis is α then the equation of the straight line is

$$x \cos \alpha + y \sin \alpha = p$$



Proof:

Let AB be the line intersects x axis at A and y axis at B.

Let OM ⊥ r AB.

Let OM = p and $\angle XOM = \alpha$.

If the intercepts are a and b then the equation of the straight line is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ....(1)$$

From right angled ΔOAM, $\frac{a}{p} = \sec\alpha \Rightarrow a = p \sec\alpha$

from ΔOBM, $\frac{b}{p} = \text{Sec}(90^o - \alpha) \Rightarrow b = p \text{ cosec } \alpha$

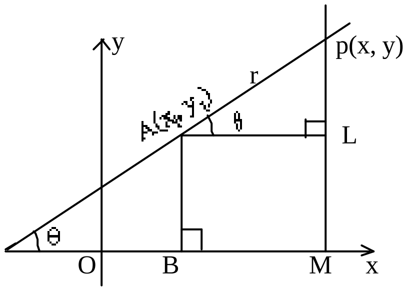
$\therefore$ (1) => $\frac{x}{p\sec\alpha} + \frac{y}{p\,\text{cosec}\alpha} = 1$

i.e., **x cos a + y sin a = p** is the equation of a straight line in normal form.

4.2.2 Symmetric form / Parametric form

If the inclination of a straight line passing through a fixed point A with x -axis is θ and any point P on the line is at a distance 'r' from A then its equation is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$



Proof:

Let $A(x_1, y_1)$ be the given point and P(x,y) be any point on the line

AP = r,

$\underline{|PAL} = \theta$

Draw PM$\perp$ OX and AL | | to x axis.

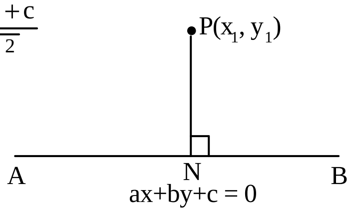
Then $\cos\theta = \frac{AL}{AP} = \frac{x - x_1}{r}$ and $\sin\theta = \frac{PL}{AP} = \frac{y - y_1}{r}$

=> $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ is the required equation.

Observation:

(i) The length of the perpendicular from $P(x_1, y_1)$ to the line ax+by+c = 0 is

$$PN = \pm \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$



(ii) The length of the perpendicular from the origin to $ax+by+c = 0$ is $\pm \frac{c}{\sqrt{a^2+b^2}}$

(iii) Equations of the bisectors of the angles between the straight lines $ax+by+c = 0$ and

$a_1x + b_1y + c_1 = 0$ are $\frac{ax+by+c}{\sqrt{a^2+b^2}} = \pm \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}$

Example 4

Find the equation of the straight line which has perpendicular distance 5 units from the origin and the inclination of perpendicular with the positive direction of x axis is 120^o

Solution:

The equation of the straight line in Normal Form is

$x \cos\alpha + y \sin\alpha = p$

Given $\alpha = 120^o$ and $p = 5$

$\therefore$ Equation of the straight line is

$x \cos 120^o + y \sin 120^o = 5$

ie $x - y\sqrt{3} + 10 = 0$

Example 5

Find the length of the perpendicular from (3,2) on the line 3x+2y+1 = 0

Solution:

Length of the perpendicular from (3,2) to the line $3x+2y+1 = 0$ is

$$\pm \frac{3(3)+2(2)+1}{\sqrt{3^2+2^2}} = \frac{14}{\sqrt{13}}$$

Example 6

Find the equation of the bisectors of the angle between 3x+4y+3 = 0 and 4x+3y+1 = 0

Solution:

The equations of the bisectors is $\frac{3x+4y+3}{\sqrt{9+16}} = \pm \frac{4x+3y+1}{\sqrt{16+9}}$

ie., $3x+4y+3 = \pm(4x+3y+1)$

ie., $x-y-2 = 0$ and $7x+7y+4 = 0$

EXERCISE 4.2

1) The portion of a straight line intercepted between the axes is bisected at the point (-3,2). Find its equation.

2) The perpendicular distance of a line from the origin is 5cm and its slope is -1. Find the equation of the line.

3) Find the equation of the straight line which passes through (2,2) and have intercepts whose sum is 9

4) Find the length of the perpendicular from the origin to the line $4x-3y+7 = 0$

5) For what value of K will the length of the perpendicular from (-1,k) to the line $5x-12y+13 = 0$ be equal to 2.

6) Find the equation of the line which has perpendicular distance 4 units from the origin and the inclination of perpendicular with +ve direction of x-axis is 135^{o}

7) Find the equation of a line which passes through the point (-2, 3) and makes an angles of 30^{o} with the positive direction of x-axis.

8) Find the equation of the bisectors of the angle between $5x+12y-7 = 0$ and $4x-3y+1 = 0$

4.3 FAMILY OF LINES

4.3.1. Intersection of two straight lines

The point of intersection of two staright lines is obtained by solving their equations.

4.3.2 Concurrent lines

Three or more straight lines are said to be concurrent when they all pass through the same point. That point is known as point of concurrency.

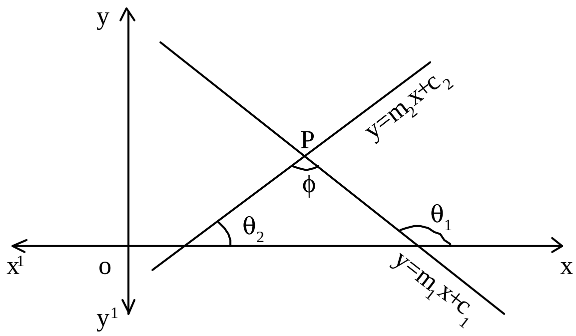
Condition for Concurrency of three lines:

$a_1x+b_1y+c_1 = 0$ ....(i) $a_2x+b_2y+c_2 = 0$ ....(ii)

$a_3x+b_3y+c_3 = 0$ ....(iii) is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

4.3.3 Angle between two straight lines



Let ϕ be the angle between the two straight lines with slopes $m_1 = \tan\theta_1$ and

$m_2 = \tan\theta_2$. Then $\tan\phi = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\therefore \phi = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Observation:

(i) If $m_1 = m_2$ the straight lines are parallel
i.e., if the straight lines are parallel then their slopes are equal.

(ii) If $m_1 m_2 = -1$ then the straight lines are $\perp$r. to each other (applicable only when the slopes are finite)
i.e., if the straight lines are perpendicular then the product of their slopes is -1.

Example 7

Show that the lines 3x+4y = 13, 2x-7y+1 = 0 and 5x-y=14 are concurrent.

Solution:

3x+4y-13 = 0
2x-7y +1 = 0 and
5x-y -14 = 0

Now,

$$\begin{vmatrix} 3 & 4 & -13 \\ 2 & -7 & 1 \\ 5 & -1 & -14 \end{vmatrix}$$

= 3(98+1) - 4 (-28-5)-13 (-2+35)

= 297 + 132-429

= 429-429 = 0

=> the given lines are concurrent

Example 8

Find the equation of a straight line through the intersection of 3x+4y = 7 and x+y-2 = 0 and having slope = 5.

Solution:

3x+4y = 7 ....(1)

x+y = 2 ....(2)

Solving (1) and (2) the point of intersection is (1, 1)

$\therefore$ (x_1, y_1) = (1, 1) and m = 5

$\therefore$ equation of the line is y-1=5(x-1)

(ie) y-1 = 5x-5

5x-y-4 = 0

Example 9

Show that the lines 5x+6y = 20 and 18x-15y = 17 are at right angles.

Solution:

The given lines are

5x+6y = 20 .... (1) and

18x-15y = 17 .... (2)

m_1 = Slope of line (1) = $-\left(\frac{5}{6}\right) = -\frac{5}{6}$

m_2 = Slope of line (2) = $-\left(\frac{18}{-15}\right) = \frac{18}{15} = \frac{6}{5}$

$m_1m_2 = \frac{-5}{6} \times \frac{6}{5} = -1$ $\therefore$ the lines are at right angles

Example 10

Find the equation of the line passing through (2,-5) and parallel to the line 4x+3y-5 = 0

Solution:

m = Slope of 4x+3y-5 = 0 is $-\frac{4}{3}$

$\therefore$ slope of the required line || to the given line = $-\frac{4}{3}$ and the line passes through $(x_1, y_1) = (2, -5)$

$\therefore$ Equation of the required line is

$$y+5 = -\frac{4}{3}(x-2) \Rightarrow 4x+3y+7 = 0$$

Example 11

Show that the triangle formed by the lnes 4x-3y-8 = 0, 3x-4y+6 = 0 and x+y-9 = 0 is an isosceles triangle

Solution:

The slope of line (1) i.e. 4x-3y-8 = 0 is $-\left(\frac{4}{-3}\right) = m_1$

$= \frac{4}{3} = m_1$

Slpe of line (2) i.e. 3x-4y+6 = 0 is $-\left(\frac{3}{-4}\right) = \frac{3}{4} = m_2$

Slope of line (3) i.e. x+y-9 = 0 is $-\left(\frac{1}{1}\right) = -1 = m_3$

If α is the angle between lines (1) and (3) then

$$\tan\alpha = \left|\frac{m_1 - m_3}{1+m_1m_3}\right| = \left|\frac{\frac{4}{3}+1}{1+\frac{4}{3}(-1)}\right| = \left|\frac{\frac{7}{3}}{\frac{-1}{3}}\right| = 7$$

$\alpha = \tan^{-1}(7)$

If β is the angle between (2) and (3)

$$\text{then } \tan\beta = \left|\frac{m_2 - m_3}{1+m_2m_3}\right| = \left|\frac{\frac{3}{4}+1}{1+\frac{3}{4}(-1)}\right| = \frac{\frac{7}{4}}{\frac{1}{4}} = 7$$

$\beta = \tan^{-1}(7)$

$\alpha = \beta$ the given triangle is an isosceles triangle.

Example 12

The fixed cost is Rs. 700 and estimated cost of 100 units is Rs. 1,800. Find the total cost y for producing x units.

Solution:

Let y = Ax + B gives the linear relation between x and y where y is the total cost, x the number of units, A and B constants.

When x = 0, fixed cost

i.e., y = 700 => O+B = 700

∴ B = 700

When x = 100, y = 1800

=> 1800 = 100A + 700

∴ A = 11

∴ The total cost y for producing x units given by the relation.

y = 11x + 700

Example 13

As the number of units produced increases from 500 to 1000 the total cost of production increases from Rs. 6,000 to Rs. 9,000. Find the relationship between the cost (y) and the number of units produced (x) if the relationship is linear.

Solution:

Let y = Ax + B where B is the fixed cost, x the number of units produced and y the total cost.

When x = 500, y = 6,000

=> 500A + B = 6,000 ----------------(1)

When x = 1000, y = 9,000

=> 1000A + B = 9,000 ----------------(2)

Solving (1) and (2) we get A = 6 and B = 3,000

∴ The linear relation between x and y is given by y = 6x + 3,000

EXERCISE 4.3

1) Show that the straight lines $4x+3y = 10$, $3x-4y = -5$ and $5x+y = 7$ are concurrent.

2) Find the value of k for which the lines $3x-4y = 7$, $4x-5y = 11$ and $2x+3y+k = 0$ are concurrent

3) Find the equation of the straight line through the intersection of the lines $x+2y+3 = 0$ and $3x+y+7 = 0$ and | | to $3y-4x = 0$

4) Find the equation of the line perpendicular to $3x+y-1 = 0$ and passing through the point of intersection of the lines $x+2y = 6$ and $y = x$.

5) The coordinates of 3 points ΔABC are A(1, 2), B(-1, -3) and C(5, -1). Find the equation of the altitude through A.

6) The total cost y of producing x units is given by the equation $3x-4y+600 = 0$ find the fixed overhead cost and also find the extra cost of producing an additional unit.

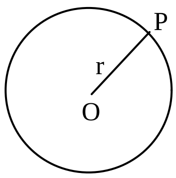
7) The fixed cost is Rs. 500 and the estimated cost of 100 units is Rs. 1,200. Find the total cost y for producing x units assuming it to be a linear function.

8) As the number of units manufactured increases from 5000 to 7000, the total cost of production increases from Rs. 26,000 to Rs. 34,000. Find the relationship between the cost (y) and the number of units made (x) if the relationship is linear.

9) As the number of units manufactured increases from 6000 to 8000, the total cost of production increases from Rs. 33,000 to Rs. 40,000. Find the relationship between the cost (y) and the number of units made (x) if the relationship is linear.

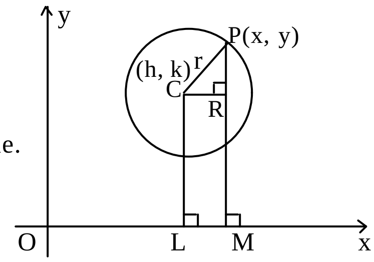
4.4 EQUATION OF CIRCLE

A circle is defined as the locus of a point which moves so that its distance from a fixed point is always a constant. The fixed point is called the centre and the constant distance is called the radius of the circle. In the fig. O is the centre and OP = r is the radius of the circle.



4.4.1 Equation of a circle whose centre and radius are given.

Solution:



Let C(h, k) be the centre and ‘r’ be the radius of the circle.

Let P(x, y) be any point on the circle.

$CP = r \Rightarrow CP^2 = r^2$

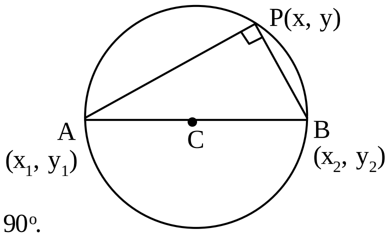
ie., $(x-h)^2 + (y-k)^2 = r^2$ is the equation of the circle.

Observation :

If the centre of the circle is at the origin (0, 0), then the equation of the circle is

$$x^2 + y^2 = r^2$$

4.4.2 The equation of a circle described on the segment joining (x_1, y_1) and (x_2, y_2) as diameter.



Let $A(x_1, y_1)$, $B(x_2, y_2)$ be the end points of the diameter of a circle whose centre is at C.

Let P(x, y) be any point on the circumference of the circle.

$\angle APB$ = angle in a semicircle = 90^o. So AP and BP are perpendicular to each other.

Slope of AP $= \dfrac{y-y_1}{x-x_1} = m_1$ (say)

Slope of BP $= \dfrac{y-y_2}{x-x_2} = m_2$ (say)

Since AP and BP are $\perp$r to each other $m_1 m_2 = -1$

$\Rightarrow \quad \dfrac{y-y_1}{x-x_1} \dfrac{y-y_2}{x-x_2} = -1$

$\Rightarrow \quad (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$ is the required equation of the circle.

4.4.3 General form of the equation of a circle

Consider the equation $x^2+y^2+2gx+2fy+c = 0$

(where g, f,c are constants) -----(1)

ie., $(x^2+2g x) + (y^2+2fy) = -c$

ie., $(x^2+2g x+g^2-g^2) + (y^2+2fy+f^2-f^2) = -c$

=> $(x^2+2gx+g^2) + (y^2+2fy+f^2) = g^2+f^2 -c$

ie., $(x+g)^2 + (y+f)^2 = g^2+f^2-c$

$[x-(-g)]^2 + [y-(-f)]^2 = \left\{\sqrt{g^2+f^2-c}\right\}^2$

Comparing this with the circle $(x-h)^2 + (y-k)^2 = r^2$ we see that (1) represents the equation to a circle with centre (-g, -f) and radius $\sqrt{g^2+f^2-c}$

Observation :

(i) It is a second degree equation in x and y.

(ii) Coefficient of x^2 = coefficient of y^2

(iii) There is no xy term

(iv) If $g^2+f^2-c > 0$, then circle is a real circle.

(v) If $g^2+f^2-c = 0$ then circle reduces to a point circle

(vi) If $g^2+f^2-c < 0$ then there is no real circle

(vii) Two or more circles having same centre are called concentric circles.

Example 14

Find the equation of the circle with centre at (3, 5) and radius 4 units

Solution:

Equation of the circle whose centre (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$

Given centre (h, k) = (3, 5) and r = 4

$\therefore$ equation of the circle is $(x-3)^2 + (y-5)^2 = 16$

=> $x^2+y^2-6x-10y+18 = 0$

Example 15

Find the equation of the circle passing through the point (1, 4) and having its centre at (2, 3)

Solution:

The distance between the centre and a point on the circumference is the radius of the circle

(ie) $r = \sqrt{(1-2)^2 + (4-3)^2} = \sqrt{1+1} = \sqrt{2}$

Given centre = (2, 3)

$\therefore$ equation of the circle is

$(x-2)^2 + (y-3)^2 = \sqrt{2}^{\,2}$

=> $x^2+y^2-4x-6y+11 = 0$

Example 16

Find the centre and radius of the circle $x^2+y^2-6x+8y-24 = 0$

Solution:

Equation of the circle is $x^2+y^2-6x+8y-24 = 0$

Identifying this with the general form of circle $x^2+y^2+2gx+2fy+c = 0$

we get $2g = -6$; $2f = 8$;

$g = -3$; $f = 4$; $c = -24$

$\therefore$ centre $= (-g, -f) = (3, -4)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{9 + 16 - (-24)} = 7$

Example 17

Find the equation of the circle when the coordinates of the end points of the diameter are (3, 2) and (-7, 8)

Solution:

The equation of a circle with end points of diamter as (x_1, y_1) and (x_2, y_2) is

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

Here $(x_1, y_1) = (3, 2)$ and

$(x_2, y_2) = (-7, 8)$

$\therefore$ equation of the circle is

$(x-3)(x+7) + (y-2)(y-8) = 0$

$x^2+y^2+4x-10y-5 = 0$

Example 18

Find the equation of the circle whose centreis (-3, 2) and circumference is 8p

Solution :

Circumference $= 2\pi r = 8\pi$

$\Rightarrow r = 4$ units

Now centre = (-3, 2) and

radius = 4

So equation of the circle is

$$(x+3)^2 + (y-2)^2 = 4^2$$

(ie) $x^2+y^2+6x-4y-3 = 0$

Example 19

Find the equation of a circle passing through the points (1, 1), (2, -1) and (2, 3)

Solution:

Let the equation of the circle be

$x^2+y^2+2gx+2fy+c = 0$ ----------------(1)

Since (1) passes through the points

(1, 1), (2, -1) and (2, 3) we get

$1+1+2g+2f+c = 0$

(ie) $2g+2f+c = -2$ -----------------(2)

$4+1+4g-2f+c = 0$

(ie) $4g-2f+c = -5$ -----------------(3)

$4+9+4g+6f+c = 0$

$4g+6f+c = -13$ -----------------(4)

Solving (2), (3) and (4) we get

$g = -\frac{7}{2}$, $f = -1$, $c = 7$. Using these values in (1) we get

$x^2+y^2-7x-2y+7 = 0$ as equation of the circle.

EXERCISE 4.4

1) Find the equation of the circle with centre at (-4, -2) and radius 6 units

2) Find the equation of the circle passing through (-2, 0) and having its centre at (2, 3)

3) Find the circumference and area of the circle $x^2+y^2-2x+5y+7=0$

4) Find the equation of the circle which is concentric with the circle $x^2+y^2+8x-12y+15 = 0$ and which passes through the point (5, 4).

5) Find the equation of the circle when the coordinates of the end points of the diameter are (2, -7) and (6, 5)

6) Find the equation of the circle passing through the points (5, 2), (2, 1) and (1, 4)

7) A circle passes through the points (4, 1) and (6, 5) and has its centre on the line $4x+y = 16$. Find the equation of the circle

8) $x+3y = 17$ and $3x-y =3$ are two diamters of a circle of radius 5 units. Find the eqution of the circle.

9) Find the equation of the circle which has its centre at (2, 3) and which passes through the intersection of the lines $3x-2y-1 = 0$ and $4x+y-27 = 0$.

4.5 TANGENTS

4.5.1 Equation of the Tangent

Let the equation of the circle be $x^2+y^2+2gx+2fy+c = 0$

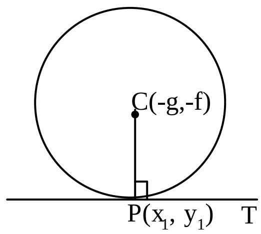
Let $P(x_1, y_1)$ be the given point on the circle and PT be the tangent at P.

The centre of the circle is C (-g, -f).

The radius through $P(x_1, y_1)$ is CP.

PT is the tangent at $P(x_1, y_1)$ and

PC is the radius



Slope of CP = $\frac{y_1+f}{x_1+g}$

$\therefore$ Slope of PT is $-\left(\frac{x_1+g}{y_1+f}\right)$ $\{\because$ PT$\perp$r to CP$\}$

$\therefore$ Equation of tangent PT at P(x_1, y_1) is $y-y_1 = -\left(\frac{x_1+g}{y_1+f}\right)(x-x_1)$

=> $yy_1 + f(y-y_1) - y_1^2+xx_1+g(x-x_1)-x_1^2 = 0$ -----------------(1)

Since (x_1, y_1) is a point on the circle

$x_1^2+y_1^2+2gx_1+2fy_1+c = 0$ -----------------(2)

(1) + (2) => $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$ is the required equation of the tangent.

Observation:

(i) From the equation of the circle, changing x^2 to xx_1, y^2 to yy_1, x to $\frac{x+x_1}{2}$ and y to $\frac{y+y_1}{2}$ and retaining the constant c we get the equation of the tangent at the point (x_1, y_1) as $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c = 0$

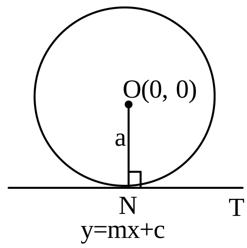
(ii) The equation of the tangent to the circle $x^2+y^2 = a^2$ at the point (x_1, y_1) is $xx_1 + yy_1 = a^2$

(iii) The length of the tangent from the point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

(iv) The point P(x_1, y_1) lies outside on or inside the circle $x^2+y^2+2gx+2fy+c = 0$ according as $x_1^2+y_1^2+2gx1+2fy_1+c \begin{matrix} \geq \\ < \end{matrix} 0$

4.5.2 Condition for the straight line y = mx+c to be tangent to the circle $x^2+y^2 = a^2$ is $c^2 = a^2 (1+m^2)$

For the line y = mx+c ie., mx-y+c = 0 to be tangent to the circle $x^2+y^2 = a^2$, the length of the perpendicular from the centre of the circle to the straight line must be equal to the radius of the circle.

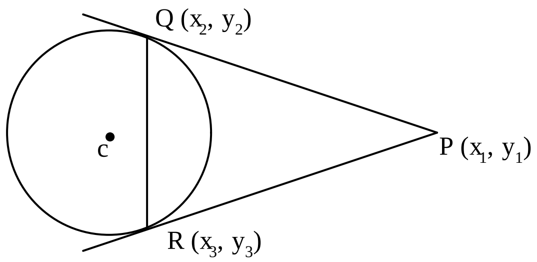


i.e., $\pm \frac{c}{\sqrt{1+m^2}} = a$

Squaring both sides we get the condition as $c^2 = a^2 (1+m^2)$

4.5.3 Chord of contact of tangents

From any point outside a circle two tangents can be drawn to the circle. The line joining the points of contacts of tangents is called the chord of contact of tangents.



The equation of chord of contact of tangents

Let the equation of the circle be $x^2+y^2+2gx+2fy+c = 0$

Let $P(x_1, y_1)$ be the given point through which the tangents PQ and PR are drawn. Then QR is the chord of contact of tangents. The equation of the tangent at Q (x_2, y_2) is

$xx_2+yy_2+g(x+x_2)+f(y+y_2)+c = 0$ ------------------(1)

The equation of tangent at $R(x_3, y_3)$ is

$xx_3+yy_3+g(x+x_3)+f(y+y_3)+c = 0$ ------------------(2)

Since these tangents pass through the point (x_1, y_1), (1) and (2) become

$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c = 0$ ------------------(3)

$x_1x_3+y_1y_3+g(x_1+x_3)+f(y_1+y_3)+c = 0$ ------------------(4)

consider $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c = 0$. This represents the equation to a straight line passing through Q and R by virtue of (3) and (4) and hence is the equation of chord of contact of the point $P(x_1, y_1)$

Example 20

Find the equation of the tangent to the circle $x^2+y^2-26x+12y+105 = 0$ at the point (7, 2)

Solution:

The equation of tangent to the circle $x^2+y^2-26x+12y+105 = 0$ at (7, 2) is

$x(7)+y(2)-13(x+7)+6(y+2)+105 = 0$

ie., $3x-4y-13 = 0$

Example 21

Find the value of p so that 3x+4y-p = 0 is a tangent to the circle $x^2+y^2-64 = 0$

Solution:

The condition for the line y = mx+c to be a tangent to the circle

$x^2+y^2 = a^2$ is $c^2 = a^2(1+m^2)$ ------------ (1)

For the given line 3x+4y = p,

$m = -\frac{3}{4}$ and $c = \frac{p}{4}$

and for the given circle $x^2+y^2 = 64$

$a = \sqrt{64} = 8$

(1) => $\left(\frac{p}{4}\right)^2 = 64[(1+(\frac{-3}{4})^2]$

$p^2 = 16x100 = 1600$

$\therefore$ $p = \pm\sqrt{1600} = \pm 40$

Example 22

Find the length of the tangent drawn from the point (-1, -3) to the circle $x^2+y^2+x+2y+6 = 0$

Solution:

Length of the tangent from (-1, -3) to the circle $x^2+y^2+x+2y+6 = 0$ is

$\sqrt{(-1)^2+(-3)^2+(-1)+2(-3)+6}$ = 3 units

EXERCISE 4.5

1) Find the equation of tangent to the circle $x^2+y^2 = 10$ at (1, 3)

2) Find the equation of tangent to the circle $x^2+y^2+2x-3y-8 = 0$ at (2, 3)

3) Find the length of the tangent from (2, -3) to the circle $x^2+y^2-8x-9y+12=0$

4) Find the condition the that line lx+my+n = 0 is a tangent to the circle $x^2+y^2 = a^2$

5) Prove that the tangents to the circle $x^2+y^2 = 169$ at (5, 12) and (12, -5) are $\perp$r to each other.

6) Find the length of the tangent from the point (-2, 3) to the circle $2x^2+2y^2 = 3$

EXERCISE 4.6

Choose the correct answer

1) If P,Q,R are points on the same line with slope of PQ = $\frac{2}{3}$, then the slope of QR is

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

2) The angle made by the line x+y+7 = 0 with the positive direction of x axis is

(a) 45^o (b) 135^o (c) 210^o (d) 60^o

3) The slope of the line 3x-5y+8 = 0 is

(a) $\frac{3}{5}$ (b) $-\frac{3}{5}$ (c) $\frac{5}{3}$ (d) $-\frac{5}{3}$

4) If the slope of a line is negative then the angle made by the line is

(a) acute (b) obtuse (c) 90^o (d) 0^o

5) The slope of a linear demand curve is

(a) positive (b) negative (c) 0 (d) ∞

6) Two lines ax+by+c = 0 and px+qy+r = 0 are $\perp$r if

(a) $\frac{a}{p} = \frac{b}{q}$ (b) $\frac{a}{b} = \frac{q}{p}$ (c) $\frac{a}{b} =- \frac{p}{q}$ (d) $\frac{a}{b} =- \frac{q}{p}$

7) Slope of the line $\perp$r to ax+by+c = 0 is

(a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$ (c) $\frac{b}{a}$ (d) $\frac{a}{b}$

8) When ax+3y+5 = 0 and 2x+6y+7 = 0 are parallel then the value of ‘a’ is

(a) 2 (b) -2 (c) 1 (d) 6

9) The value of ‘a’ for which 2x+3y-7 = 0 and 3x+ay+5 = 0 are $\perp$r is

(a) 2 (b) -2 (c) 3 (d) -3

10) The centre of the circle $x^2+y^2+6y-9 = 0$ is

(a) (0, 3) (b) (0, -3) (c) (3, 0) (d) (-3, 0)

11) The equation of the circle with centre at (0, 0) and radius 3 units is

(a) $x^2+y^2 = 3$ (b) $x^2+y^2 = 9$ (c) $x^2+y^2 = \sqrt{3}$ (d) $x^2+y^2 = 3\sqrt{3}$

12) The length of the diameter of a circle with centre (1, 2) and passing through the point (5, 5) is

(a) 5 (b) $\sqrt{45}$ (c) 10 (d) $\sqrt{50}$

13) If (1, -3) is the centre of the circle $x^2+y^2+ax+by+9 = 0$, its radius is

(a) $\sqrt{10}$ (b) 1 (c) 5 (d) $\sqrt{19}$

14) The area of the circle $(x-2)^2 + (y-4)^2 = 25$ is

(a) 25 (b) 5 (c) 10 (d) $25\,\overline{\Lambda}$

15) The equation of tangent at (1, 2) to the circle $x^2+y^2 = 5$ is

(a) $x+y = 5$ (b) $x+2y = 5$ (c) $x-y = 5$ (d) $x-2y = 5$

16) The length of tangent from (3, 4) to the circle $x^2+y^2-4x+6y-1 = 0$ is

(a) 7 (b) 6 (c) 5 (d) 8

17) If $y = 2x+c$ is a tangent to the circle $x^2+y^2 = 5$ then the value of c is

(a) $\pm\sqrt{5}$ (b) ± 25 (c) ± 5 (d) ± 2

TRIGONOMETRY 5

The Greeks and Indians saw trigonometry as a tool for the study of astronomy. Trigonometry, derived from the Greek words "Trigona" and "Metron", means measurement of the three angles of a triangle. This was the original use to which the subject was applied. The subject has been considerably developed and it has now wider application and uses.

The first significant trigonometry book was written by Ptolemy around the second century A.D. George Rheticus (1514-1577) was the first to define trigonometric functions completely in terms of right angles. Thus we see that trigonometry is one of the oldest branches of Mathematics and a powerful tool in higher mathematics.

Let us recall some important concepts in trigonometry which we have studied earlier.

Recall

1. Measurement of angles (Sexagesimal system)

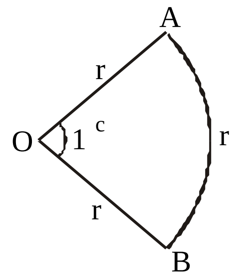
a) one right angle $= 90^o$

b) one degree (1^o) = 60' (Minutes)

c) one minute (1') = 60'' (Seconds)

2. Circular Measure (or) Radian measure

Radian : A radian is the magnitude of the angle subtended at the centre by an arc of a circle equal in length to the radius of the circle. It is denoted by 1^c. Generally the symbol "c" is omitted.



π radian $= 180^o$, 1 radian $= 57^o\ 17'\ 45''$

Radians	$\frac{p}{6}$	$\frac{p}{4}$	$\frac{p}{3}$	$\frac{p}{2}$	π	$3\frac{p}{2}$	2π
Degrees	30^o	45^o	60^o	90^o	180^o	270^o	360^o

3. Angles may be of any magnitude not necessarily restricted to 90°. An angle is positive when measured anti clockwise and is negative when measured clockwise.

5.1 TRIGONOMETRIC IDENTITIES

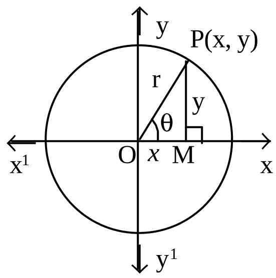


Fig 5.1

Consider the circle with centre at the origin **O** (0, 0) and radius r units. Let **P**(x, y) be any point on the circle. Draw PM ⊥ to OX. Now, ΔOMP is a right angled triangle with one vertex at the origin of a coordinate system and one vertex on the positive X-axis. The other vertex is at P, a point on the circle.

Let $\underline{|XO}P = \theta$

From ΔOMP, OM = x = side adjacent to θ

MP = y = side opposite θ

OP = r = length of the hypotenuse of ΔOMP

Now, we define

Sine function : **sinq** $= \dfrac{\textit{length of the side opposite } \mathbf{q}}{\textit{length of the hypotenuse}} = \dfrac{y}{r}$

Cosine function : **cosq** $= \dfrac{\textit{length of the side adjacent to è}}{\textit{length of the hypotenuse}} = \dfrac{x}{r}$

Tangent function : **tan q** $= \dfrac{\textit{length of the side opposite è}}{\textit{length of the side adjacent to è}} = \dfrac{y}{x}$

[redacted]

the sine, cosine and tangent functions respectively.

i.e. **cosecq** $= \dfrac{1}{\sin\theta} = \dfrac{r}{y}$

secq $= \dfrac{1}{\cos\theta} = \dfrac{r}{x}$

cotq $= \dfrac{1}{\tan\theta} = \dfrac{x}{y}$

Observation :

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$; $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(ii) If the circle is a unit circle then r = 1.

$\therefore \quad \text{Sin}\,\theta = y \quad ; \quad \text{cosec}\,\theta = \frac{1}{y}$

$\cos\theta = x \quad ; \quad \sec\theta = \frac{1}{x}$

(iii)

Function	**Cofunction**
sine	cosine
tangent	cotangent
secant	cosecant

(iv) $(\sin\theta)^2$, $(\sec\theta)^3$, $(\tan\theta)^4$, ... and in general $(\sin\theta)^n$ are written as $\sin^2\theta$, $\sec^3\theta$, $\tan^4\theta$, ... $\sin^n\theta$ respectively. But $(\cos x)^{-1}$ is not written as $\cos^{-1}x$, since the meaning for $\cos^{-1}x$ is entirely different. (being the angle whose cosine is x)

5.1.1 Standard Identities

(i) $\sin^2 q + \cos^2 q = 1$

Proof: From right angled triangle OMP, (fig 5.1)

we have $x^2 + y^2 = r^2$

$\cos^2\theta + \sin^2\theta = 1 \qquad (\because r = 1)$

(ii) $1 + \tan^2 q = \sec^2 q$

Proof : $1 + \tan^2\theta = 1 + \frac{y^2}{x^2}$

$$= \frac{x^2+y^2}{x^2} = \frac{r^2}{x^2} = \frac{1}{x^2} = \left(\frac{1}{x}\right)^2 = \sec^2\theta$$

(iii) $1 + \cot^2 q = \text{cosec}^2 q$

Proof : $1 + \cot^2\theta = 1 + \frac{x^2}{y^2}$

$$= \frac{y^2+x^2}{y^2} = \frac{r^2}{y^2} = \frac{1}{y^2} = \left(\frac{1}{y}\right)^2 = \text{cosec}^2\theta$$

Thus, we have

(i)	**$\sin^2 q + \cos^2 q = 1$**
(ii)	**$1 + \tan^2 q = \sec^2 q$**
(iii)	**$1 + \cot^2 q = \text{cosec}^2 q$**

Example 1

Show that $\cos^4A - \sin^4A = 1-2\sin^2A$

Solution:

$$\begin{aligned}\cos^4A-\sin^4A &= (\cos^2A + \sin^2A)(\cos^2A-\sin^2A)\\ &= \cos^2A-\sin^2A\\ &= 1-\sin^2A-\sin^2A\\ &= 1-2\sin^2A\end{aligned}$$

Example 2

Prove that $(\sin A+\cos A)(1-\sin A\cos A) = \sin^3A + \cos^3A$

Solution:

$$\begin{aligned}\text{R.H.S.} &= \sin^3A + \cos^3A\\ &= (\sin A+\cos A)(\sin^2A + \cos^2A - \sin A\cos A)\\ &= (\sin A + \cos A)(1-\sin A\cos A) = \text{L.H.S.}\end{aligned}$$

Example 3

Show that $\sec^4A - 1 = 2\tan^2A + \tan^4A$

Solution :

$$\begin{aligned}\text{L.H.S.} &= \text{Sec}^4A-1\\ &= (\sec^2A+1)(\sec^2A-1)\\ &= (1+\tan^2A+1)(1+\tan^2A-1)\\ &= (2+\tan^2A)\tan^2A\\ &= 2\tan^2A + \tan^4A = \text{R.H.S.}\end{aligned}$$

Example 4

Prove that $$\frac{1+\tan^2A}{1+\cot^2A} = \frac{\sin^2A}{\cos^2A} = \tan^2A$$

Solution:

$$\frac{1+\tan^2A}{1+\cot^2A} = \frac{\sec^2A}{\text{cosec}^2A} = \frac{\left(\frac{1}{\cos^2A}\right)}{\left(\frac{1}{\sin^2A}\right)} = \frac{\sin^2A}{\cos^2A} = \tan^2A$$

Example 5

Prove that $$\frac{1}{\sec q - \tan q} = \sec q + \tan q$$

Solution:

L.H.S. = $\frac{1}{\sec\theta - \tan\theta}$

Mutiply numerator and denominator each by $(\sec\theta + \tan\theta)$

$$= \frac{\sec\theta + \tan\theta}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}$$

$$= \frac{\sec\theta + \tan\theta}{\sec^2\theta - \tan^2\theta} = \sec\theta + \tan\theta. = \text{R.H.S}$$

Example 6

Prove that $\frac{\cot A + \tan B}{\cot B + \tan A}$ **= cotA tanB**

Solution :

L.H.S. = $\frac{\cot A + \tan B}{\cot B + \tan A} = \frac{\cot A + \tan B}{\frac{1}{\tan B} + \frac{1}{\cot A}}$

$$= \frac{\cot A + \tan B}{\left(\frac{\cot A + \tan B}{\cot A \tan B}\right)}$$

= cotA tanB = R.H.S.

Example 7

Prove that $(\sin q + \text{cosec}\, q)^2 + (\cos q + \sec q)^2 = \tan^2 q + \cot^2 q + 7$

Solution :

L.H.S. $= (\sin\theta + \text{cosec}\,\theta)^2 + (\cos\theta + \sec\theta)^2$

$= \sin^2\theta + \text{cosec}^2\theta + 2\sin\theta\,\text{cosec}\,\theta + \cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta$

$= (\sin^2\theta + \cos^2\theta) + (1+\cot^2\theta) + 2 + (1+\tan^2\theta) + 2$

$= 1 + 6 + \tan^2\theta + \cot^2\theta$

$= \tan^2\theta + \cot^2\theta + 7$ = R.H.S.

Example 8

Prove that $(1+\cot A+\tan A)(\sin A-\cos A) = \frac{\sec A}{\text{cosec}^2 A} - \frac{\text{cosec} A}{\sec^2 A}$

Solution:

L.H.S. = (1+cotA+tanA)(sinA-cosA)

= sinA - cosA + cotAsinA - cotA cosA + tanAsinA - tanA cosA

$= \sin A - \cos A + \cos A - \frac{\cos^2 A}{\sin A} + \frac{\sin^2 A}{\cos A} - \sin A$

$= \frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A}$

$= \frac{\sec A}{\text{cosec}^2 A} - \frac{\text{cosec} A}{\sec^2 A}$

Recall

q	**0°**	**30°**	**45°**	**60°**	**90°**
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Example 9

If $A = 45^o$, verify that (i) $\sin 2A = 2\sin A \cos A$ (ii) $\cos 2A = 1-2\sin^2 A$

Solution:

(i) L.H.S. $= \sin 2A$

$= \sin 90^o = 1$

R.H.S. $= 2\sin A \cos A = 2\sin 45^o \cos 45^o$

$= 2 \left(\frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}\right)$

$= 1$

Hence verified.

(ii) L.H.S. $= \cos 2A = \cos 90^o = 0$

R.H.S. $= 1 - 2\sin^2 A = 1-2\sin^2 45^o$

$= 1 - 2 \left(\frac{1}{\sqrt{2}}\right)^2$

$= 1-1 = 0$

Hence verified.

Example 10

Prove that $4\cot^2 45^o - \sec^2 60^o + \sin^3 30^o = \frac{1}{8}$

Solution:

$$\text{L.H.S.} = 4\cot^2 45^\circ - \sec^2 60^\circ + \sin^3 30^\circ$$

$$= 4(1)^2 - (2)^2 + \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{8} = \text{R.H.S.}$$

EXERCISE 5.1

1) If $a\sin^2\theta + b\cos^2\theta = c$, show that $\tan^2\theta = \frac{c-b}{a-c}$

2) Prove that $\frac{1}{\cot A + \tan A} = \sin A \cos A$

3) Prove that $\frac{1-\tan A}{1+\tan A} = \frac{\cot A - 1}{\cot A + 1}$

4) Prove that $\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = 2\sec^2\theta$

5) Prove that $\operatorname{cosec}^4 A - \operatorname{cosec}^2 A = \cot^2 A + \cot^4 A$

6) Prove that $\frac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \frac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2\sec^2 A$

7) Prove that $(1+\cot A - \operatorname{cosec} A)(1+\tan A + \sec A) = 2$

8) Prove that $\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A$

9) Show that $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \operatorname{cosec}\theta\sec\theta$

10) Show that $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$

11) If $A = 30^\circ$, verify that
   (i) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
   (ii) $\sin 2A = 2\sin A \cos A$
   (iii) $\cos 3A = 4\cos^3 A - 3\cos A$
   (iv) $\sin 3A = 3\sin A - 4\sin^3 A$
   (v) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

12) Find the value of $\frac{4}{3}\cot^2 30^\circ + 2\sin^2 60^\circ - 2\operatorname{cosec}^2 60^\circ - \frac{3}{4}\tan^2 30^\circ$

13) Find $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^3 30^\circ$

14) Find $\cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3}$

15) If $\sec A + \tan A = \frac{3}{2}$, prove that $\tan A = \frac{5}{12}$

16) If $4\tan A = 3$, show that $\frac{5\sin A - 2\cos A}{\sin A + \cos A} = 1$

17) If $a\cos\theta + b\sin\theta = c$ and $b\cos\theta - a\sin\theta = d$ show that $a^2 + b^2 = c^2 + d^2$

18) If $\tan\theta = \frac{1}{\sqrt{7}}$ find the value of $\frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta}$

19) If $\sec^2\theta = 2 + 2\tan\theta$, find $\tan\theta$

20) If $x = \sec\theta + \tan\theta$, then show that $\sin\theta = \frac{x^2 - 1}{x^2 + 1}$

5.2 SIGNS OF TRIGONOMETRIC RATIOS

5.2.1 Changes in signs of the Trigonometric ratios of an angle θ as θ varies from 0^o to 360^o

Consider the circle with centre at the origin **O**(0,0) and radius r units Let P(x,y) be any point on the circle.

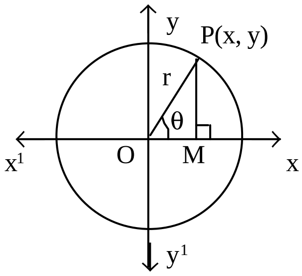


Fig 5.2(a)

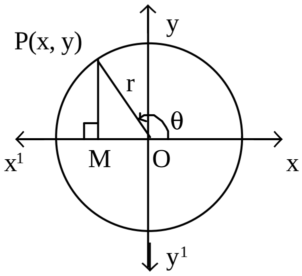


Fig 5.2(b)

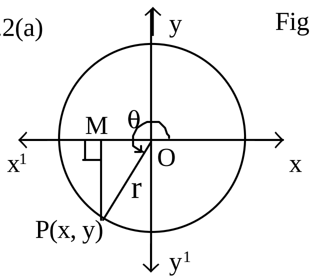


Fig 5.2(c)

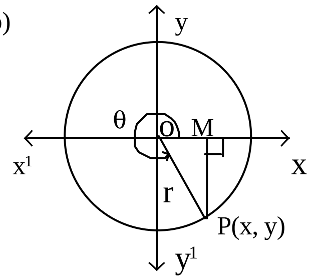


Fig 5.2(d)

Let the revolving line OP=r, makes an angle θ with OX

Case (1) Let q be in the first quadrant i.e. $0^o < q < 90^o$

From fig 5.2(a) the coordinates of P, both x and y are *positive.* Therefore all the trigonometric ratios are *positive.*

Case (2) Let q be in the second quadrant i.e. $90^o < q < 180^o$

From fig 5.2(b) the x coordinate of P is negative and y coordinate of P is *positive*. Therefore sin θ is *positive*, cos θ is *negative* and tan θ is *negative.*

Case (3) Let q be in the third quadrant i.e. $180^o < q < 270^o$

From fig 5.2(c), both x and y coordinates of P are *negative.* Therefore sinθ and cosθ are negative and tanθ is *positive.*

Case (4) Let q be in the fourth quadrant i.e. $270^o < q < 360^o$

From fig 5.2(d), x coordinate of P is *positive* and y coordinate of P is *negative.* Therefore sinθ and tanθ are *negative* and cosθ is *positive.*

Thus we have

Quadrant	sinq	cosq	tanq	cosecq	secq	cotq
I	+	+	+	+	+	+
II	+	-	-	+	-	-
III	-	-	+	-	-	+
IV	-	+	-	-	+	-

A simple way of remembering the signs is by refering this chart:

S	A
T	C

A → In I quadrant All trigonometric ratios are positive

S → In II quadrant Sinθ and Cosecθ alone are *positive* and all others are *negative*.

T → In III quadrant Tanθ and Cotθ alone are *positive* and all others are *negative*.

C → In IV quadrant Cosθ and Secθ alone are *positive* and all others are *negative*.

5.2.2 Determination of the quadrant in which the given angle lies

Let θ be less than 90^o Then the angles:

$(90^o-\theta)$ lies in first quadrant	$(270^o-\theta)$ lies in third quadrant
$(90^o+\theta)$ lies in second quadrant	$(270^o+\theta)$ lies in fourth quadrant
$(180^o-\theta)$ lies in second quadrant	$(360^o-\theta)$ lies in fourth quadrant
$(180^o+\theta)$ lies in third quadrant	$(360^o+\theta)$ lies in first quadrant

Observation :

(i) 90^o is taken to lie either in I or II quadrant.

(ii) 180^o is taken to lie either in II or III quadrant

(iii) 270^o is taken to lie either in III or IV quadrant

(iv) 360^o is taken to lie either in IV or I quadrant

Example 11

Determine the quadrants in which the following angles lie

(i) 210^o (ii) 315^o (iii) 745^o

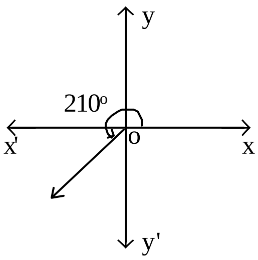


Fig. 5.3(a)

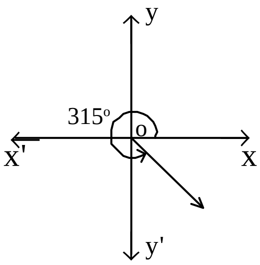


Fig. 5.3(b)

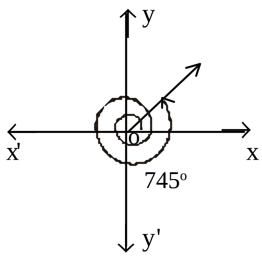


Fig. 5.3(c)

From fig 5.3(a)
$210^o = 180^o + 30^o$
This is of the form
$180^o + \theta^o$
$\therefore$ 210^o lies in
Third quadrant.

From fig 5.3(b)
$315^o = 270^o + 45^o$
This is of the form
$270^o + \theta^o$.
$\therefore$ 315^o lies in
Fourth quadrant

From fig 5.3(c)
we see that
745^o = Two complete rotations plus 25^o
$745^o = 2x360^o + 25^o$
$\therefore$ 745^o lies in *First quadrant.*

5.2.3 Trigonometric ratios of angles of any magnitude

In order to find the values of the trigonometric functions for the angles more than 90^o, we can follow the useful methods given below.

(i) Determine the quadrant in which the given angle lies.

(ii) Write the given angle in the form $\mathbf{k}\frac{\mathbf{p}}{2} \pm \mathbf{q}$, **k** is a positive integer

(iii) Determine the sign of the given trigonometric function in that particular quadrant using the chart:

S	A
T	C

(iv) If k is even, trigonometric form of allied angle equals the same function of θ

(v) If k is odd, trigonometric form of the allied angle equals the cofunction of θ and vice versa

Observation:

From fig. 5.4 "- θ^o" is same as (360^o - θ^o).

∴ sin(-θ)= sin(360^o-θ) = -sinθ

cos(-θ) = cosθ

tan(-θ) = -tanθ

cosec(-θ) = -cosecθ

sec (-θ) = secθ

cot(-θ) = -cotθ.

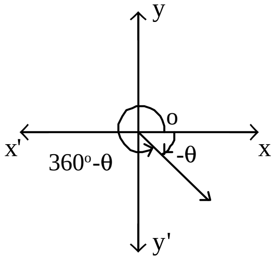


Fig 5.4

Angles / Functions	-q	90^o - q	90^o+q	180^o - q	180^o+q	270^o - q	270^o+q	360^o - q	360^o+q
sine	-sin θ	cosθ	cosθ	sinθ	-sin θ	-cosθ	-cosθ	-sin θ	sinθ
cos	cosθ	sinθ	-sin θ	-cosθ	- cos θ	-sin θ	sinθ	cosθ	cosθ
tan	-tanθ	cotθ	-cotθ	-tanθ	tanθ	cotθ	-cotθ	-tanθ	tanθ
cosec	-cosecθ	secθ	secθ	cosecθ	-cosecθ	-secθ	-secθ	-cosecθ	cosecθ
sec	secθ	cosecθ	-cosecθ	-secθ	-secθ	-cosecθ	cosecθ	secθ	secθ
cot	-cotθ	tanθ	-tanθ	-cotθ	cotθ	tanθ	-tanθ	-cotθ	cotθ

Example 12

Find the values of the following

(i) sin (120^o) **(ii) tan(-210^o)** **(iii) sec(405^o)**

(iv) cot(300^o) **(v) cos(-330^o)** **(vi) cosec(135^o)** **vii) tan 1145^o**

Solution:

(i) $120^o = 90^o+30^o$

It is of the form $90^o+\theta^o$ ∴120^o is in second quadrant

sin(120^o) = sin(90^o+30^o)

$= \cos 30^o = \frac{\sqrt{3}}{2}$

(ii) $\tan(-210^o) = -\tan(210^o)$

$= -\tan(180^o+30^o)$

$= -\tan30^o = -\frac{1}{\sqrt{3}}$

(iii) $\sec(405^o) = \sec[360^o+45^o] = \sec45^o = \sqrt{2}$

(iv) $\cot(300^o) = \cot(360^o-60^o)$

$= -\cot60^o = -\frac{1}{\sqrt{3}}$

(v) $\cos(-330^o) = \cos(330^o)$

$= \cos(270^o+60^o)$

$= \sin60^o = \frac{\sqrt{3}}{2}$

(vi) $\text{cosec}(135^o) = \text{cosec}(90^o+45^o)$

$= \sec45^o = \sqrt{2}$

(vii) $\tan(1145^o) = \tan(12x90^o + 65^o)$

$= \tan65^o = \tan(90^o-25^o) = \cot25^o$

Example 13

Find the following : (i) sin843^o (ii) cosec(-757^o) (iii) cos(-928^o)

Solution:

(i) $\sin843^o = \sin(9x90^o+33^o)$

$= \cos33^o$

(ii) $\text{cosec}(-757^o) = -\text{cosec}(757^o)$

$= -\text{cosec}(8x90^o+37^o) = -\text{cosec}\ 37^o$

(iii) $\cos(-928^o) = \cos(928^o)$

$= \cos(10x90^o + 28^o) = -\cos28^o$

Observation :

Angles / Functions	**180^o**	**270^o**	**360^o**
sin	0	-1	0
cos	-1	0	1
tan	0	-∞	0
cosec	∞	-1	∞
sec	-1	∞	1
cot	∞	0	∞

EXERCISE 5.2

1) Prove that : $\sin 420^o \cos 390^o - \cos(-300^o)\sin(-330^o) = \frac{1}{2}$

2) If A, B, C are the angles of a triangle, show that

(i) $\sin(A+B) = \sin C$ (ii) $\cos(A+B) + \cos C = 0$ (iii) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$

3) If A lies between 270^o and 360^o and $\cot A = -\frac{24}{7}$, find cosA and cosecA.

4) If $\sin\theta = \frac{11}{12}$, find the value of :

$\sec(360^o-\theta)\tan(180^o-\theta) + \cot(90^o+\theta)\sin(270^o+\theta)$

5) Find the value of $\sin 300^o \tan 330^o \sec 420^o$

6) Simplify $\dfrac{\sin\left(\frac{\pi}{2}-A\right)\cos(\pi-A)\tan(\pi+A)}{\sin\left(\frac{\pi}{2}+A\right)\sin(\pi-A)\tan(\pi-A)}$

7) Prove that $\sin 1140^o \cos 390^o - \cos 780^o \sin 750^o = \frac{1}{2}$

8) Evaluate the following (i) $\sec 1327^o$ (ii) $\cot(-1054^o)$

5.3 COMPOUND ANGLES

In the previous section we have found the trigonometric ratios of angles such as $90^o \pm \theta$, $180^o \pm \theta$, ... which involve only single angles. In this section we shall express the trigonometric ratios of compound angles.

When an angle is made up of the algebraic sum of two or more angles, it is called compound angle. For example $A \pm B$, $A+B+C$, $A-2B+3C$, etc are compound angles.

5.3.1 Addition and Subtraction Formulae

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$

5.3.2 Prove goemetrically :
cos(A-B) = cosAcosB + sinAsinB

Proof: Consider the unit circle whose centre is at the origin O(0,0).

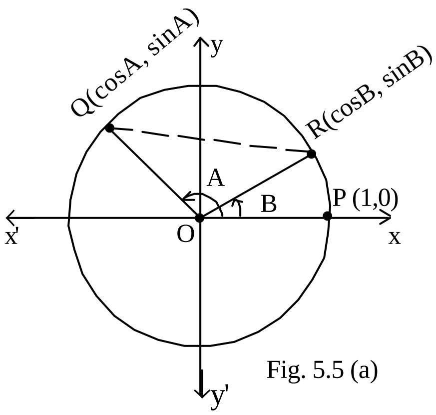


Fig. 5.5 (a)

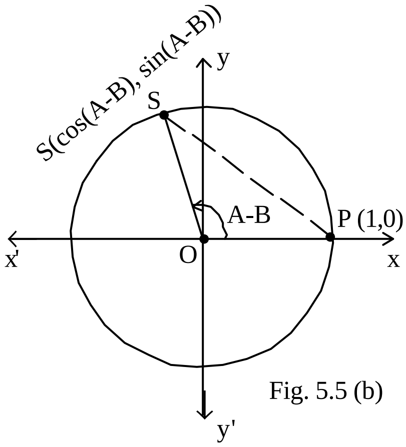


Fig. 5.5 (b)

Let P(1,0) be a point on the unit circle

Let $\angle$**A** and $\angle$**B** be any two angles in standard position

Let Q and R be the points on the terminal side of angles A and B, respectively.

From fig 5.5(a) the co-ordinates of Q and R are found to be, Q (cosA, sinA) and R (cosB, sinB). Also we have $\angle$**ROQ** = A-B.

Now move the points Q and R along the circle to the points S and P respectively in such a way that the distance between P and S is equal to the distance between R and Q. Therefore we have from Fig. 5.5(b); $\angle$POS = $\angle$ROQ = A-B; and

S[cos(A-B), sin(A-B)]
Also, $PS^2 = RQ^2$

By the distance formula, we have

$\{\cos(A-B)-1\}^2 + \sin^2(A-B) = (\cos A-\cos B)^2 + (\sin A-\sin B)^2$

$\cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B) = \cos^2 A-2\cos A\cos B +$
$\cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$

$2 - 2\cos(A-B) \qquad = 2 - (2\cos A\cos B + 2\sin A\sin B)$

$\therefore$ **cos(A-B) = cosAcosB + sinAsinB.**

Corollary (i)

$\cos(A+B) = \cos[A-(-B)]$

$= \cos A\cos(-B) + \sin A\sin(-B)$

$= \cos A\cos B + \sin A\{-\sin B\}$

\ **cos(A+B) = cosAcosB - sinAsinB**

Corollary (ii)

$\sin(A+B) = \cos[\frac{\pi}{2} - (A+B)]$

$= \cos[(\frac{\pi}{2} - A) - B]$

$= \cos(\frac{\pi}{2} - A)\cos B + \sin(\frac{\pi}{2} - A)\sin B$

\ **sin(A+B) = sinAcosB + cosAsinB**

Corollary (iii)

$\sin(A-B) = \sin[A+(-B)]$

$= \sin A\cos(-B) + \cos A\sin(-B)$

\ **sin(A-B) = sinAcosB - cosAsinB**

Corollary (iv)

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)}$$

$$\backslash \quad \mathbf{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

Corollary (v)

$\tan(A-B) = \tan[A+(-B)]$

$$= \frac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)}$$

$$\therefore \quad \mathbf{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

Example 14

Find the values of the following : (i) cos15^o (ii) tan75^o

Solution:

(i) $\cos 15^o = \cos(45^o - 30^o)$

$= \cos 45^o \cos 30^o + \sin 45^o \sin 30^o$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\tan 75^o = \tan(45^o + 30^o)$

$$= \frac{\tan 45^o + \tan 30^o}{1 - \tan 45^o \tan 30^o}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Example 15

If A and B be acute angles with cosA = $\frac{5}{13}$ and sinB = $\frac{3}{5}$ find cos(A-B)

Solution:

Given $\cos A = \frac{5}{13}$ $\therefore$ $\sin A = \sqrt{1 - \frac{25}{169}}$

$$= \sqrt{\frac{169-25}{169}} = \frac{12}{13}$$

Given $\sin B = \frac{3}{5}$ $\therefore$ $\cos B = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

$\therefore$ $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$= \frac{5}{13} \cdot \frac{4}{5} + \frac{12}{13} \cdot \frac{3}{5} = \frac{56}{65}$$

Example 16

If sinA = $\frac{1}{3}$, cosB = - $\frac{3}{4}$ and A and B are in second quadrant, then find (i) sin(A+B), (ii) cos(A+B), (iii) tan(A+B) and determine the quadrant in which A+B lies.

Solution:

$$\cos A = \sqrt{1 - \sin^2 A} = -\frac{2\sqrt{2}}{3}$$

(since A is in second quadrant cosA is *negative*)

$$\sin B = \sqrt{1 - \cos^2 B}$$

$$\sin B = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

(Since B is in second quadrant sinB is *positive*)

$$\therefore \tan A = \frac{\sin A}{\cos A} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{-2\sqrt{2}}{3}\right)} = -\frac{\sqrt{2}}{4}$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{\left(\frac{\sqrt{7}}{4}\right)}{\left(\frac{-3}{4}\right)} = \frac{-\sqrt{7}}{3}$$

$$\text{Sin}(A+B) = \sin A\cos B + \cos A\sin B$$

$$= \frac{1}{3}\left(\frac{-3}{4}\right) + \left(\frac{-2\sqrt{2}}{3}\right)\left(\frac{\sqrt{7}}{4}\right)$$

$$= -\frac{1}{4} - \frac{2\sqrt{14}}{12} = -\left(\frac{1}{4} + \frac{2\sqrt{14}}{12}\right)$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$= \left(\frac{-2\sqrt{2}}{3}\right)\left(\frac{-3}{4}\right) - \frac{1}{3}\,\frac{\sqrt{7}}{4}$$

$$= \frac{6\sqrt{2}-\sqrt{7}}{12} \text{ is } \textit{positive}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$= \frac{\frac{-1}{4}\sqrt{2} - \frac{1}{3}\sqrt{7}}{1 - \left\{\left(\frac{-1}{4}\sqrt{2}\right)\left(\frac{-1}{3}\sqrt{7}\right)\right\}}$$

$$= -\left(\frac{3\sqrt{2}+4\sqrt{7}}{12-\sqrt{14}}\right)$$

Since sin(A+B) is negative and cos(A+B) is positive (A+B) must be in the ***fourth quadrant.***

Example 17

If A+B = 45^o prove that (1+tanA)(1+tanB) = 2 and deduce the value of tan22$\frac{1}{2}^o$

Solution:

Given A+B = 45^o

$\therefore$ tan(A+B) = tan45^o = 1

$$\frac{\tan A+\tan B}{1-\tan\ \tan B} = 1$$

=> tanA + tanB + tanAtanB = 1

Adding 1 to both sides

1 + tanA+tanB+tanAtanB = 1+1 = 2

i.e. (1+tanA) (1+tanB) = 2 --------------------(1)

Putting A = B = 22$\frac{1}{2}^o$ in (1), we get $(1 + \tan 22\frac{1^o}{2})^2 = 2$

$$\Rightarrow 1 + \tan 22\frac{1}{2}^o = \underline{+}\sqrt{2}$$

$\therefore$ $1+\tan 22\frac{1}{2}^o = \sqrt{2}$ (since $22\frac{1}{2}^o$ is an angle in I quadrant,

$1+\tan 22\frac{1}{2}^o$ is positive)

$\therefore$ $\tan 22\frac{1}{2}^o = \sqrt{2} - 1$

Example 18

Prove that cos(60^o+A) cos(30^o-A) - sin(60^o-A) sin(30^o-A) = 0

Proof :

Let $\alpha = 60^o+A$

$\beta = 30^o-A$

Then the given problem is of the form cos($\alpha+\beta$)

i.e. cos[(60^o+A)+(30^o-A)]

= cos(60^o+30^o)

= cos90^o

= 0

EXERCISE 5.3

1) Show that

(i) $\sin(A+B)\sin(A-B) = \sin^2A - \sin^2B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2A - \sin^2B$

2) Prove the following : $\text{Sin}(A-45^o) + \text{Cos}(45^o+A) = 0$

3) Prove that $\tan75^o + \cot75^o = 4$

4) If $\tan\theta = \frac{1}{2}$, $\tan\phi = \frac{1}{3}$, then show that $\theta+\phi = \frac{\pi}{4}$

5) Find the values of : (i) $\tan105^o$ (ii) $\sec105^o$.

6) Prove that $\frac{\sin(A-B)}{\sin A\sin B} + \frac{\sin(B-C)}{\sin B\sin C} + \frac{\sin(C-A)}{\sin C\sin A} = 0$

7) Prove that $\frac{\cos(x+y)}{\cos(x-y)} = \frac{1-\tan x\tan y}{1+\tan x\tan y}$

8) If $\cos A = -\frac{12}{13}$, $\cos B = \frac{24}{25}$, A is obtuse and B is acute angle find

(i) sin(A+B) (ii) cos(A-B)

9) Prove that $\sin A + \sin(120^o+A) + \sin(240^o+A) = 0$

10) Show that $\cot15^o + \cot75^o + \cot135^o = 3$

11) If tanA + tanB = a; cotA + cotB = b, show that $\cot(A+B) = \frac{1}{a} - \frac{1}{b}$

5.3.3 Multiple angles

In this section, we shall obtain formulae for the trigonometric functions of 2A and 3A. There are many aspects of integral calculus where these formulae play a key role.

We know that sin(A+B) = sinAcosB + cosAsinB and When **A=B**,

sin2A = sinAcosA + cosAsinA

$\therefore$ **sin2A = 2sinAcosA**

Similarly, if we start with

cos(A+B) = cosAcosB - sinAsinB and when **A=B** we obtain

cos2A = cosAcosA - sinAsinA

cos2A = cos²A - sin²A

Also, cos2A = $\cos^2A - \sin^2A$

$= (1-\sin^2 A) - \sin^2 A$

$= \mathbf{1-2\sin^2 A}$

$\cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1-\cos^2 A)$

$= \mathbf{2\cos^2 A - 1}$

We know that, $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A \tan B}$. When **A=B** we obtain

$$\mathbf{\tan 2A = \frac{2\tan A}{1-\tan^2 A}}$$

Also we can prove the following

(i) $\mathbf{\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}}$

(ii) $\mathbf{\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}}$

Proof: (i) we have

$\sin 2A = 2\sin A \cos A$

$= 2\tan A \cos^2 A$

$= \dfrac{2\tan A}{\sec^2 A} = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) we have

$\cos 2A = \cos^2 A - \sin^2 A$

$= \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$ ($\because$ $1=\cos^2 A+\sin^2 A$)

$\mathbf{\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}}$

Observation :

(i) $\sin^2 A = \dfrac{1-\cos 2A}{2}$

(ii) $\cos^2 A = \dfrac{1+\cos 2A}{2}$

(iii) $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

5.3.4 To express sin3A, cos3A and tan3A interms of A

(i) $\sin 3A = \sin(2A+A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A \cos^2 A + (1-2\sin^2 A) \sin A$

$= 2\sin A(1-\sin^2 A) + (1-2\sin^2 A) \sin A$

$$\mathbf{\sin 3A = 3\sin A - 4\sin^3 A}$$

(ii) $\cos 3A = \cos(2A+A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A-1) \cos A - 2\sin^2 A \cos A$

$= (2\cos^2 A-1) \cos A - 2(1-\cos^2 A)\cos A$

$$\mathbf{\cos 3A = 4\cos^3 A - 3\cos A}$$

(iii) $\tan 3A = \tan(2A+A)$

$$= \frac{\tan 2A + \tan A}{1-\tan A \tan 2A}$$

$$= \frac{\frac{2\tan A}{1-\tan^2 A} + \tan A}{1-\tan A\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{2\tan A + \tan A\left(1-\tan^2 A\right)}{1-\tan^2 A - 2\tan^2 A}$$

$$\mathbf{\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}}$$

5.3.5 Sub multiple angle

$\sin A = \sin(2\frac{A}{2}) = 2\sin\frac{A}{2} \cos\frac{A}{2}$

$\cos A = \cos(2\frac{A}{2}) = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1$

$= 1 - 2\sin^2\frac{A}{2}$

$$\tan A = \tan(2\frac{A}{2}) = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

Further,

(i) **sinA** $= \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(ii) **cosA** $= \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(iii) $\sin^2\dfrac{A}{2} = \dfrac{1-\cos A}{2}$

(iv) $\cos^2\dfrac{A}{2} = \dfrac{1+\cos A}{2}$

(v) $\tan^2\dfrac{A}{2} = \dfrac{1-\cos A}{1+\cos A}$

Example 19

Prove that $\dfrac{\sin 2A}{1-\cos 2A}$ **= cotA**

Solution:

L.H.S. $= \dfrac{\sin 2A}{1-\cos 2A} = \dfrac{2\sin A\cos A}{2\sin^2 A}$

$= \dfrac{\cos A}{\sin A}$

$= \cot A$ = R.H.S.

Example 20

Find the values of

(i) $\sin 22\frac{1}{2}^{o}$ **(ii)** $\cos 22\frac{1}{2}^{o}$ **(iii)** $\tan 22\frac{1}{2}^{o}$

Solution:

(i) $\sin^2\dfrac{A}{2} = \dfrac{1-\cos A}{2}$

$\sin^2\dfrac{45}{2} = \dfrac{1-\cos 45^{o}}{2} = \dfrac{1-\frac{1}{\sqrt{2}}}{2} = \dfrac{2-\sqrt{2}}{4}$

$\therefore \sin 22\frac{1}{2}^{o} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

(ii) $\cos^2 \frac{A}{2} = \frac{1+\cos A}{2}$

$\therefore \cos 22\frac{1}{2}^o = \frac{\sqrt{2+\sqrt{2}}}{2}$

(iii) $\tan^2 \frac{A}{2} = \frac{1-\cos A}{1+\cos A}$

$\tan^2 \frac{45}{2} = \frac{1-\cos 45^o}{1+\cos 45^o}$

$= \frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$= (\sqrt{2}-1)^2$

$\therefore \tan 22\frac{1}{2}^o = \sqrt{2} - 1$

Example 21

If $\tan A = \frac{1}{3}$, $\tan B = \frac{1}{7}$ prove that $2A+B = \frac{p}{4}$

Solution:

$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2} = \frac{3}{4}$

$\tan(2A+B) = \frac{\tan 2A + \tan B}{1-\tan 2A \tan B} = \frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\ \frac{1}{7}} = 1$

$\Rightarrow 2A+B = \frac{\pi}{4}$ ($\because \tan 45^o = 1$)

Example 22

If $\tan A = \frac{1-\cos B}{\sin B}$, provethat $\tan 2A = \tan B$, where A and B are acute angles.

Solution:

Given $\tan A = \frac{1-\cos B}{\sin B}$

$$= \frac{2\sin^2\frac{B}{2}}{2\sin\frac{B}{2}\cos\frac{B}{2}} = \tan\frac{B}{2}$$

$\therefore \quad \tan A = \tan\frac{B}{2}$

$\Rightarrow \quad A = \frac{B}{2}$

i.e. $\quad 2A = B$

$\therefore \tan 2A = \tan B$

Example 23

Show that $\sin 20^o \sin 40^o \sin 60^o \sin 80^o = \frac{3}{16}$

Solution

L.H.S. $= \sin 60^o.\sin 20^o.\sin(60^o-20^o).\sin(60^o+20^o)$

$$= \frac{\sqrt{3}}{2}\sin 20^o\,[\sin^2 60^o - \sin^2 20^o]$$

$$= \frac{\sqrt{3}}{2}\sin 20^o\,[\,\frac{3}{4} - \sin^2 20^o)$$

$$= \frac{\sqrt{3}}{2}\;\frac{1}{4}\,[3\sin 20^o - 4\sin^3 20^o]$$

$$= \frac{\sqrt{3}}{2}\;\frac{1}{4}\;\sin 60^o$$

$$= \frac{\sqrt{3}}{2}\;\frac{1}{4}\;\frac{\sqrt{3}}{2} = \frac{3}{16} = \text{R.H.S.}$$

Example 24

Find the values of $\sin 18^o$ and $\cos 36^o$

Solution:

Let $\theta = 18^o$, then $5\theta = 5x18 = 90^o$

$3\theta + 2\theta = 90^o$

$\therefore 2\theta = 90^o - 3\theta$

$\therefore \quad \sin 2\theta = \sin(90^o - 3\theta) = \cos 3\theta$

$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$ divide by $\cos\theta$ on both sides

$2\sin\theta = 4\cos^2\theta - 3 \qquad (\because \cos\theta \neq 0)$

$$2\sin\theta = 4(1-\sin^2\theta)-3$$
$$2\sin\theta = 1-4\sin^2\theta$$

$\therefore 4\sin^2\theta + 2\sin\theta - 1 = 0$, which is a quadratic equation in $\sin\theta$.

$$\therefore \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8}$$

$$= \frac{-1 \pm \sqrt{5}}{4}$$

since $\theta = 18^o$, which is an acute angle, $\sin\theta$ is +ve

$$\therefore \sin 18^o = \frac{\sqrt{5}-1}{4}$$

$$\cos 36^o = 1-2\sin^2 18^o \qquad = 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}$$

Example 25

Prove that $\mathbf{\dfrac{\cos 3A}{\cos A} + \dfrac{\sin 3A}{\sin A} = 4\cos 2A.}$

$$\text{L.H.S.} = \frac{\cos 3A}{\cos A} + \frac{\sin 3A}{\sin A}$$

$$= \frac{\sin A \cos 3A + \cos A \sin 3A}{\cos A \sin A} = \frac{\sin(A+3A)}{\sin A \cos A}$$

$$= \frac{\sin 4A}{\sin A \cos A}$$

$$= \frac{2\sin 2A \cos 2A}{\sin A \cos A}$$

$$= \frac{2 \,.\, 2\sin A \cos A \cos 2A}{\sin A \cos A}$$

$$= 4\cos 2A = \text{R.H.S.}$$

Example 26

Prove that $\mathbf{\dfrac{1+\sin q - \cos q}{1+\sin q + \cos q} = \tan \dfrac{q}{2}}$

Solution:

$$\text{L.H.S.} = \frac{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \left(1-2\sin^2\frac{\theta}{2}\right)}{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + 2\cos^2\frac{\theta}{2} - 1}$$

$$= \frac{2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}$$

$$= \tan\frac{\theta}{2} = \text{R.H.S.}$$

EXERCISE 5.4

1) Prove that $\tan A + \cot A = 2\operatorname{cosec} 2A$

2) Prove that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$

3) If $\tan\theta = \frac{1}{7}$, $\tan\phi = \frac{1}{3}$, then prove that $\cos 2\theta = \sin 4\phi$

4) If $2\cos\theta = x + \frac{1}{x}$ then prove that

(i) $\cos 2\theta = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$

(ii) $\cos 3\theta = \frac{1}{2}\left(x^3 + \frac{1}{x^3}\right)$

5) Prove that $\frac{\sin 3A + \sin^3 A}{\cos^3 A - \cos 3A} = \cot A$

6) Show that $\frac{1+\sin 2A}{1-\sin 2A} = \tan^2(45^\circ + A)$

7) If $\tan\frac{A}{2} = t$, then prove that

(i) $\sin A + \tan A = \frac{4t}{1-t^4}$

(ii) $\sec A + \tan A = \frac{(1+t)^2}{1-t^2}$

8) Show that $\cos^2 36^\circ + \sin^2 18^\circ = \frac{3}{4}$

[redacted]

10) Prove that $\frac{1-\cos 3A}{1-\cos A} = (1+2\cos A)^2$

11) Prove that $\frac{\cos 2A}{1+\sin 2A} = \tan(45^\circ - A)$

12) Prove that $(\sin \frac{A}{2} - \cos \frac{A}{2})^2 = 1-\sin A$

13) Show that $\frac{1-\tan^2 (45^o - \theta)}{1+\tan^2 (45^o - \theta)} = \sin 2\theta$

14. If $\sin A = \frac{3}{5}$ find sin3A, cos3A and tan3A

15. Show that $\frac{\cos 3A}{\cos A} = 2\cos 2A-1$

16. Prove that $\sec^2 A(1+\sec 2A) = 2\sec 2A$

5.3.6 Transformation of products into sums or differences

we have

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ ....(1)

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ ....(2)

(1)+(2), gives

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$ ....(a)

(1)-(2), gives

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$ ....(b)

Also we have

$\cos(A+B) = \cos A\cos B-\sin A\sin B$ ....(3)

$\cos(A-B) = \cos A\cos B+\sin A\sin B$ ....(4)

(3)+(4), gives

$\cos(A+B)+\cos(A-B) = 2\cos A\cos B$ ....(c)

(4)-(3), gives

$\cos(A-B) - \cos(A+B) = 2\sin A.\sin B$ ....(d)

Example 27

Express the following as sum or difference:

(i) 2sin3q cosq (ii) 2cos2q cosq (iii) 2sin3x sinx

(iv) cos9q cos7q (v) $\cos 7\frac{A}{2} \cos 9\frac{A}{2}$ (vi) cos5q sin4q

vii) 2cos11A sin13A

Solution:

(i) $2\sin 3\theta \cos\theta = \sin(3\theta+\theta) + \sin(3\theta-\theta)$
$= \sin 4\theta+\sin 2\theta$

(ii) $2\cos 2\theta \cos\theta = \cos(2\theta+\theta) + \cos(2\theta-\theta)$
$= \cos 3\theta + \cos\theta$

(iii) 2sin3x sinx $= \cos(3x-x) - \cos(3x+x)$
$= \cos 2x - \cos 4x$

(iv) $\cos 9\theta \cos 7\theta$ $= \frac{1}{2}[\cos(9\theta+7\theta) + \cos(9\theta-7\theta)]$

$= \frac{1}{2}[\cos 16\theta + \cos 2\theta]$

(v) $\cos 7\frac{A}{2} \cos 9\frac{A}{2}$ $= \frac{1}{2}[\cos(7\frac{A}{2}+9\frac{A}{2}) + \cos(7\frac{A}{2}-9\frac{A}{2})]$

$= \frac{1}{2}[\cos 8A + \cos(-A)]$

$= \frac{1}{2}[\cos 8A + \cos A]$

(vi) $\cos 5\theta \sin 4\theta$ $= \frac{1}{2}[\sin 9\theta - \sin\theta]$

(vii) 2cos11A sin13A $= \sin(11A+13A) - \sin(11A-13A)$
$= \sin 24A + \sin 2A$

Example 28

Show that $4\cos a \cos(120^o-a)\cos(120^o+a) = \cos 3a$.

Solution:

L.H.S. $= 2\cos\alpha \; 2\cos(120^o-\alpha)\cos(120^o+\alpha)$
$= 2\cos\alpha.\{\cos(120^o-\alpha+120^o+\alpha) + \cos(120^o-\alpha-120^o-\alpha)\}$
$= 2\cos\alpha\{\cos 240^o + \cos(-2\alpha)\}$
$= 2\cos\alpha\{\cos 240^o + \cos 2\alpha\}$
$= 2\cos\alpha\{-\frac{1}{2} + 2\cos^2\alpha - 1\}$
$= 4\cos^3\alpha - 3\cos\alpha$
$= \cos 3\alpha$ = R.H.S.

5.3.7 Transformation of sums or differences into products

Putting C = A+B and D = A-B in (a), (b), (c) and (d) of 5.3.6 We get

(i) **sinC + sinD** $= 2\sin\frac{C+D}{2} \cos\frac{C-D}{2}$

(ii) **sinC - sinD** $= 2\cos\frac{C+D}{2} \sin\frac{C-D}{2}$

(iii) **cosC + cosD** $= \mathbf{2\cos\frac{C+D}{2}\ \cos\frac{C-D}{2}}$

(iv) **cosC - cosD** $= \mathbf{-2\sin\frac{C+D}{2}\ \sin\frac{C-D}{2}}$

Example 29

Express the following as product.

(i) sin7A+sin5A **(ii) sin5q-sin2q** **(iii) cos6A+cos8A**

(iv) cos2a-cos4a **(v) cos10º-cos20º** **(vi) cos55º+cos15º**

(vii) cos65º+sin55º

Solution:

(i) sin7A + sin5A $= 2\sin\left(\frac{7A+5A}{2}\right)\cos\left(\frac{7A-5A}{2}\right)$

$= 2\sin 6A \cos A$

(ii) $\sin 5\theta - \sin 2\theta$ $= 2\cos\left(\frac{5\theta+2\theta}{2}\right)\sin\left(\frac{5q-2q}{2}\right)$

$= 2\cos\frac{7\theta}{2}\ \sin\frac{3\theta}{2}$

(iii) cos6A + cos8A $= 2\cos\left(\frac{6A+8A}{2}\right)\cos\left(\frac{6A-8A}{2}\right)$

$= 2\cos 7A \cos(-A) = 2\cos 7A \cos A$

(iv) $\cos 2\alpha - \cos 4\alpha$ $= 2\sin\left(\frac{4\alpha+2\alpha}{2}\right)\sin\left(\frac{4\alpha-2\alpha}{2}\right)$

$= 2\sin 3\alpha.\sin\alpha$

(v) $\cos 10^o - \cos 20^o$ $= 2\sin\left(\frac{20^o+10^o}{2}\right)\sin\left(\frac{20^o-10^o}{2}\right)$

$= 2\sin 15^o \sin 5^o$

(vi) $\cos 55^o + \cos 15^o$ $= 2\cos\left(\frac{55^o+15^o}{2}\right)\cos\left(\frac{55^o-15^o}{2}\right)$

$= 2\cos 35^o \cos 20^o$

(vii) $\cos 65^o + \sin 55^o$ $= \cos 65^o + \sin(90^o-35^o)$

$= \cos 65^o + \cos 35^o$

$= 2\cos\left(\frac{65^o+35^o}{2}\right)\cos\left(\frac{65^o-35^o}{2}\right)$

$= 2\cos 50^o \cos 15^o$

Example 30

Prove that $(\cos a + \cos b)^2 + (\sin a - \sin b)^2 = 4\cos^2\left(\frac{a+b}{2}\right)$

$\cos\alpha + \cos\beta \quad = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ ....(1)

$\sin\alpha - \sin\beta \quad = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ ....(2)

$(1)^2 + (2)^2$

$(\cos\alpha+\cos\beta)^2+(\sin\alpha-\sin\beta)^2$

$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right) + 4\cos^2\left(\frac{\alpha+\beta}{2}\right).\sin^2\left(\frac{\alpha-\beta}{2}\right)$

$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\left\{\cos^2\left(\frac{\alpha-\beta}{2}\right) + \sin^2\left(\frac{\alpha-\beta}{2}\right)\right\}$

$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right)$

Example 31

Show that $\cos^2 A + \cos^2(60^o+A) + \cos^2(60^o-A) = \frac{3}{2}$

$\cos^2 A \quad = \frac{1+\cos 2A}{2}$ ....(1)

$\cos^2(60^o+A) \quad = \frac{1+\cos 2(60^o+A)}{2}$ ....(2)

$\cos^2(60^o-A) \quad = \frac{1+\cos 2(60^0-A)}{2}$ ....(3)

(1)+(2)+(3)

$\cos^2 A + \cos^2(60^o+A) + \cos^2(60^o-A)$

$= \frac{1}{2}[3 + \cos 2A + \{\cos(120^o+2A) + \cos(120^o-2A)\}]$

$= \frac{1}{2}[3 + \cos 2A + 2\cos 120^o.\cos 2A]$

$= \frac{1}{2}[3 + \cos 2A + 2(-\frac{1}{2})\cos 2A$

$= \frac{3}{2}$

EXERCISE 5.5

1) Express in the form of a sum or difference

(i) $\sin\frac{A}{4}\ \sin\frac{3A}{4}$ (ii) sin(B+C).sin(B-C)

(iii) $\sin(60^o+A).\sin(120^o+A)$ (iv) $\cos\frac{5A}{3}\cos\frac{4A}{3}$

2) Express in the form of a product:

(i) $\sin52^o-\sin32^o$ (ii) cos6A-cos2A (iii) $\sin50^o+\cos80^o$

3) Prove that $\cos20^o.\cos40^o\cos60^o\cos80^o = \frac{1}{16}$

4) Prove that sin(A-B) sinC + sin(B-C) sinA + sin(C-A).sinB = 0

5) Prove that $\frac{\cos B-\cos A}{\sin A-\sin B} = \tan\frac{A+B}{2}$

6) Prove that $\sin50^o - \sin70^o + \cos80^o = 0$

7) Prove that $\cos18^o + \cos162^o + \cos234^o + \cos306^o = 0$

8) Prove that $(\cos\alpha-\cos\beta)^2 + (\sin\alpha-\sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$

9) Prove that $(\cos\alpha+\cos\beta)^2 + (\sin\alpha+\sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

10) Prove that $\cos40^o + \cos80^o + \cos160^o = 0$

11) Prove that $\cos20^o + \cos100^o + \cos140^o = 0$

12) If sinA + sinB = x, cosA + cosB = y, show that $\sin(A+B) = \frac{2xy}{x^2+y^2}$

13) Prove that $\frac{\cos2A-\cos3A}{\sin2A+\sin3A} = \tan\frac{A}{2}$

5.4 TRIGONOMETRIC EQUATIONS

Equations involving trigonometric functions are known as *trigonometric equations*.

For example: $2\sin\theta=1$; $\sin^2\theta+\cos\theta-3=0$; $\tan^2\theta-1=0$ etc;

The values of 'θ' which satisfy a trigonometric equation are known as *solution of the equation.*

5.4.1 Principal solution

Among all solutions, the solution which is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for *sine ratio*, in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for *tan ratio* and in $[0, \pi]$ for *cosine ratio* is the *principal solution.*

Example 32

Find the principal solution of the following equations:

(i) cosq = $-\frac{\sqrt{3}}{2}$ (ii) tanq = $\sqrt{3}$ (iii) sinq = $-\frac{1}{2}$

Solution:

(i) $\cos\theta = -\frac{\sqrt{3}}{2} < 0$

$\therefore$ θ lies in second or third quadrant.

But $\theta \in [0, \pi]$.Hence the principal solution is in second quadrant.

$\therefore \cos\theta = -\frac{\sqrt{3}}{2} = \cos(180^o - 30^o)$

$= \cos 150^o$

$\therefore$ Principal solution θ is $5\frac{\pi}{6}$

(ii) $\tan\theta = \sqrt{3} > 0$

$\therefore$ θ is in the first or third quadrant

$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore$ The solution is in first quadrant

$\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$

$\frac{p}{3} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore$ Principal solution is $\theta = \frac{\pi}{3}$

(iii) $\sin\theta = -\frac{1}{2} < 0$

$\therefore$ θ lies in third or fourth quadrant

$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\therefore$ The principal solution is in fourth quadrant and $\theta = -\frac{\pi}{6}$

5.4.2 General solutions of the Trigonometric equations

(i) If **sinq = sina** ; $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$
then **q = np + (-1)^n a; nÎZ**

(ii) If **cosq = cosa;** $0 \le \alpha \le \pi$
then **q = 2np ± a; nÎZ**

(iii) If **tanq = tana;** $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$
then **q = np + a; nÎZ**

Example 33

Find the general solution of the following equations.

(i) sinq = $\frac{1}{2}$ **(ii) cosq = $-\frac{1}{2}$** **(iii) tanq = $\sqrt{3}$**

(iv) tanq = -1 **(v) sinq = $-\frac{\sqrt{3}}{2}$.**

Solution:

(i) $\sin\theta = \frac{1}{2}$ => $\sin\theta = \sin 30^o = \sin\frac{\pi}{6}$

This is of the form $\sin\theta = \sin\alpha$

where $\alpha = \frac{\pi}{6}$

$\therefore$ the general solution is $\theta = n\pi + (-1)^n.\alpha$; $n \in Z$

i.e. $\theta = n\pi + (-1)^n.\frac{\pi}{6}$; $n \in Z$

(ii) $\cos\theta = -\frac{1}{2}$ => $\cos\theta = \cos 120^o = \cos\frac{2\pi}{3}$

$\therefore \theta = 2n\pi \pm 2\frac{\pi}{3}$; $n \in Z$.

(iii) $\tan\theta = \sqrt{3}$ => $\tan\theta = \tan 60^o = \tan\frac{\pi}{3}$

$\therefore \theta = n\pi + \frac{\pi}{3}$; $n \in Z$

(iv) $\tan\theta = -1 \Rightarrow \tan\theta = \tan 135^o = \tan\frac{3\pi}{4}$

$\Rightarrow \theta = n\pi + \frac{3\pi}{4}$; $n \in Z$

((v) $\sin\theta = -\frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \sin(-\frac{\pi}{3})$

$\Rightarrow \theta = n\pi + (-1)^n.(-\frac{\pi}{3})$; $n \in Z$

ie $\theta = n\pi - (-1)^n.\frac{\pi}{3}$; $n \in Z$

Example 34

Find the general solution of the following

(i) $\sin^2 q = 1$ (ii) $\cos^2 q = \frac{1}{4}$ (iii) $\text{cosec}^2 q = \frac{4}{3}$

(iv) $\tan^2 q = \frac{1}{3}$

Solution:

(i) $\sin^2\theta = 1 \therefore \sin\theta = \pm 1 \Rightarrow \sin\theta = \sin(\pm\frac{p}{2})$

$\therefore \theta = n\pi + (-1)^n (\pm\frac{p}{2})$

i.e. $\theta = n\pi \pm \frac{p}{2}$; $n \in Z$.

(ii) $\cos^2\theta = \frac{1}{4} \Rightarrow 1 - \sin^2\theta = \frac{1}{4} \Rightarrow \sin^2\theta = \frac{3}{4} \therefore \sin\theta = \pm\frac{\sqrt{3}}{2}$

$\therefore \sin\theta = \sin(\pm\frac{p}{3})$

$\Rightarrow \theta = n\pi \pm \frac{p}{3}$; $n \in Z$.

(iii) $\text{cosec}^2\theta = \frac{4}{3}$ or $\text{cosec}\theta = \pm\frac{2}{\sqrt{3}} \Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2}$

$\therefore \theta = n\pi \pm \frac{\pi}{3}$; $n \in Z$.

(iv) $\tan^2\theta = \frac{1}{3}$ or $\tan\theta = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta = \tan(\pm 30^o)$

$\Rightarrow \tan\theta = \tan(\pm \frac{p}{6})$

$\therefore$ General solution is $\theta = n\pi \pm \frac{p}{6}$; $n \in Z$

EXERCISE 5.6

1) Find the principal solution of the following:

(i) $\text{cosec}\theta = 2$ (ii) $\sec\theta = -\frac{2}{\sqrt{3}}$ (iii) $\cos\theta = -\frac{1}{\sqrt{2}}$

(iv) $\tan\theta = \frac{1}{\sqrt{3}}$ (v) $\cot\theta = -1$ (vi) $\sin\theta = \frac{1}{\sqrt{2}}$

2) Solve:

(i) $\cot^2\theta = \frac{1}{3}$ (ii) $\sec^2\theta = 4$ (iii) $\text{cosec}^2\theta = 1$

(iv) $\tan^2\theta = 3$.

5.5 INVERSE TRIGONOMETRIC FUNCTIONS

The quantities such as $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc., are known as *inverse trigonometric functions.*

If $\sin\theta = x$, then $\theta = \sin^{-1}x$. Here the symbol $\sin^{-1}x$ denotes the angle whose sine is x.

The two quantities $\sin\theta = x$ and $\theta = \sin^{-1}x$ are identical. (Note that, $\sin^{-1}x \neq (\sin x)^{-1}$)

For example, $\sin\theta = \frac{1}{2}$ is same as $\theta = \sin^{-1}(\frac{1}{2})$

Thus we can write $\tan^{-1}(1) = \frac{\pi}{4}$, $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$ etc.

5.5.1 Important properties of inverse trigonometric functions

1. **(i) $\sin^{-1}(\sin q) = q$** **(iv) $\text{cosec}^{-1}(\text{cosec}\, q) = q$**

(ii) $\cos^{-1}(\cos q) = q$ **(v) $\sec^{-1}(\sec q) = q$**

(iii) $\tan^{-1}(\tan q) = q$ **(vi) $\cot^{-1}(\cot q) = q$**

2. **(i)** $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$ **(iv)** $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$ **(v)** $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$ **(vi)** $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$

3. **(i)** $\sin^{-1}(-x) = -\sin^{-1} x$ **(ii)** $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$ **(iv)** $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x.$

4. **(i)** $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

(ii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Example 35

Evaluate the following

(i) $\sin\left(\cos^{-1}\frac{3}{5}\right)$ **(ii)** $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Solution:

(i) Let $\cos^{-1}\frac{3}{5} = \theta$ ....(1)

$\therefore \cos\theta = \frac{3}{5}$

We know, $\sin\theta = \sqrt{1-\cos^2\theta} = \frac{4}{5}$

Now, $\sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\theta$, using (1)

$= \frac{4}{5}$

(ii) Let $\tan^{-1}\left(\frac{3}{4}\right) = \theta$ ....(1)

$\therefore \tan\theta = \frac{3}{4}$

We can prove $\tan\theta = \frac{3}{4} \Rightarrow \cos\theta = \frac{4}{5}$

$\cos(\tan^{-1}\frac{3}{4}) = \cos\theta$ using (1)

$= \frac{4}{5}$

Example 36

(i) Prove that: $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right)$

(ii) $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$

Proof:

(i) $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left[\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\frac{1}{13}}\right]$

$= \tan^{-1}\left[\frac{20}{90}\right] = \tan^{-1}\left(\frac{2}{9}\right)$

(ii) Let $\cos^{-1}\left(\frac{4}{5}\right) = \theta$

$\therefore \cos\theta = \frac{4}{5} \Rightarrow \tan\theta = \frac{3}{4}$

$\therefore \cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

$\therefore \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}$

$= \tan^{-1}\left[\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\frac{3}{5}}\right]$

$= \tan^{-1}\left(\frac{27}{11}\right)$

Example 37

Prove that

(i) $\sin^{-1}(3x-4x^3) = 3\sin^{-1}x$ **(ii)** $\cos^{-1}(4x^3-3x) = 3\cos^{-1}x$

Proof:

i) $\sin^{-1}(3x-4x^3)$

Let $x = \sin\theta \quad \therefore \theta = \sin^{-1}x$.

$3x-4x^3 = 3\sin\theta - 4\sin^3\theta = \sin 3\theta$ ....(1)

Now, $\sin^{-1}(3x-4x^3) = \sin^{-1}(\sin 3\theta)$, using (1)

$= 3\theta$

$= 3\sin^{-1}x$

ii) $\cos^{-1}(4x^3-3x)$

Let $x = \cos\theta \therefore \theta = \cos^{-1}x$

$4x^3 - 3x = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$ ....(1)

Now, $\cos^{-1}(4x^3-3x) = \cos^{-1}(\cos 3\theta)$, using (1)

$= 3\theta$

$= 3\cos^{-1}x$

Example 38

Solve: $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

Solution:

L.H.S. $= \tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right)$

$$= \tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x^2-1}{x^2-4}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{(x-1)(x+2)+(x+1)(x-2)}{x^2-4}}{\frac{x^2-4-x^2+1}{x^2-4}}\right] = \tan^{-1}\left[\frac{2x^2-4}{-3}\right]$$

Since, $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, we have

$$\tan^{-1}\left(\frac{2x^2-4}{-3}\right) = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{2x^2-4}{-3}\right) = \tan^{-1}(1)$$

Hence $\frac{2x^2-4}{-3} = 1$

$=> 2x^2 - 4 = -3$

$=> 2x^2 - 1 = 0$

$=> x^2 = \frac{1}{2}$

$\therefore x = \pm\frac{1}{\sqrt{2}}$

EXERCISE 5.7

1) Show that $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\left[\frac{xy-1}{x+y}\right]$

2) Show that $\tan^{-1}x + \tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4}$

3) Prove that $\tan^{-1}(5) - \tan^{-1}(3) + \tan^{-1}\left(\frac{7}{9}\right) = n\pi + \frac{\pi}{4}$; $n \in Z$

4) Prove that $2\tan^{-1}x = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]$ [Hint: Put $x=\tan\theta$]

5) Prove that $2\sin^{-1}x = \sin^{-1}[2x\sqrt{1-x^2}]$ [Hint Put $x=\sin\theta$]

6) Solve : $\tan^{-1}2x+\tan^{-1}3x = \frac{\pi}{4}$

7) Solve : $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}(\frac{4}{7})$

8) Prove that $\cos^{-1}(\frac{4}{5}) + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$

9) Evaluate $\cos[\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}]$ [Hint: Let $A = \sin^{-1}\frac{3}{5}$

$B = \sin^{-1}\frac{5}{13}$]

10) Prove that $\tan^{-1}(\frac{4}{3}) - \tan^{-1}(\frac{1}{7}) = \frac{\pi}{4}$

EXERCISE 5.8

Choose the correct answer:

1) If p cosecθ = cot45^o then p is

(a) cos45^o (b) tan45^o (c) sin45^o (d) sinθ

2) $\sqrt{1-\cos^2\theta}$ x $\sqrt{1-\sin^2\theta}$ - $\left(\frac{\cos\theta}{\text{cosec}\,\theta}\right)$ = ...

(a) 0 (b) 1 c) $\cos^2\theta - \sin^2\theta$ d) $\sin^2\theta - \cos^2\theta$.

3) $(\sin 60^o + \cos 60^o)^2 + (\sin 60^o - \cos 60^o)^2$ = ...

(a) 3 (b) 1 (c) 2 (d) 0

4) $\frac{1}{\sec 60^o - \tan 60^o}$ = ...

(a) $\frac{\sqrt{3}+2}{2\sqrt{3}}$ (b) $\frac{\sqrt{3}-2}{2\sqrt{3}}$ (c) $\frac{1+\sqrt{3}}{2}$ (d) $\frac{1-\sqrt{3}}{2}$

5) If $x = a\cos^3\theta$; $y = b\sin^3\theta$ then $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}}$ is equal to

(a) $2\cos^3\theta$ (b) $3b\sin^3\theta$ (c) 1 (d) $ab\sin^2\theta\cos^2\theta$

6) The value of $\frac{1}{\sec(-60^o)}$ is

(a) $\frac{1}{2}$ (b) -2 (c) 2 (d) $-\frac{1}{2}$

7) Sin(90^o+θ) sec(360^o-θ) =

(a) cosecθ (b) 1 (c) -1 (d) cosθ

8) sec(θ-π) =

(a) secθ (b) -cosecθ (c) cosecθ (d) -secθ

9) When sinA = $\frac{1}{\sqrt{2}}$, between 0^o and 360^o the two values of A are

(a) 60^o and 135^o (b) 135^o and 45^o (c) 135^o and 175^o (d) 45^o and 225^o

10) If $\cos(2n\pi + \theta) = \sin\alpha$ then

(a) $\theta - \alpha = 90^o$ (b) $\theta = \alpha$ (c) $\theta + \alpha = 90^o$ (d) $\alpha - \theta = 90^o$

11) $\frac{\tan 15^o - \tan 75^o}{1 + \tan 15^o \tan 75^o}$ is equal to

(a) $\frac{1+\sqrt{3}}{1-\sqrt{3}}$ (b) $\frac{1+2\sqrt{3}}{1-2\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $\sqrt{3}$

12) The value of tan 435° is

(a) $\frac{1+\sqrt{3}}{1-\sqrt{3}}$ (b) $\frac{1+\sqrt{3}}{\sqrt{3}-1}$ (c) $\frac{\sqrt{3}-1}{1-\sqrt{3}}$ (d) 1

13) The value of cos9°cos6° - sin9° sin6° is

(a) 0 (b) $\frac{\sqrt{3}+1}{4}$ (c) sin75° (d) sin15°

14) $\tan(\frac{\pi}{4}+x)$ is

(a) $\frac{1+\tan x}{1-\tan x}$ (b) 1+tanx (c) -tanx (d) $\tan\frac{\pi}{4}$

[redacted] [redacted]

(a) 0 (b) 1 (c) ∞ (d) -1

16) If sinA = 1, then sin2A is equal to

(a) 2 (b) 1 (c) 0 (d) -1

17) The value of sin54° is

(a) $\frac{1-\sqrt{5}}{4}$ (b) $\frac{\sqrt{5}-1}{4}$ (c) $\frac{\sqrt{5}+1}{4}$ (d) $\frac{-\sqrt{5}-1}{4}$

18) $\frac{1-\cos 15^{o}}{1+\cos 15^{o}}$ = ...

a) sec30° (b) $\tan^2\left(\frac{15}{2}\right)$ (c)tan30° (d) $\tan^2 7\frac{1}{2}^{o}$

19) $\sin^2 40^o - \sin^2 10^o$ =

(a) sin80° (b) $\frac{\sqrt{3}}{2}$ (c) $\sin^2 30^o$ (d) $\frac{\sin 50^0}{2}$

20) The value of $\frac{3\tan\frac{\pi}{4}-\tan^3\frac{\pi}{4}}{1-3\tan^2\frac{\pi}{4}}$ is equal to

(a) -1 (b) 1 (c) 0 (d) ∞

21) The value of 4sin18°.cos36° is

(a) 0 (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $-\frac{\sqrt{3}}{2}$

22) The principal solution of cosx = 1 is

(a) x = 1 (b) x = 0 (c) x = 0° (d) x = 360°

23) If sinx = 0, then one of the solutions is

(a) $x = 3\frac{\pi}{2}$ (b) $x = 4\frac{\pi}{3}$ (c) $x = 5\pi$ (d) $x = 5\frac{\pi}{2}$

24) If cosx = 0, then one of the soutions is

(a) $x = 2\pi$ (b) $x = 14\frac{\pi}{3}$ (c) $x = 21\frac{\pi}{2}$ (d) $x = 180^o$

25) If tanx = 0; then one of the solutions is

(a) $x = 0^o$ (b) $x = \frac{\pi}{2}$ (c) $x = \frac{p}{18}$ (d) $x = -2\frac{\pi}{3}$

26) If sinx = k, where; $-1 \leq k \leq 1$ then the principal solution of x may lie in

(a) $[0, \frac{\pi}{2}]$ (b) $[-\infty, -\pi]$ (c) (0,1) (d) $(\frac{p}{2}, \infty)$

27) If cosx = k, where $-1 \leq k \leq 1$ then the principal solution of x may lie in

(a) $[-\infty, -\frac{\pi}{2}]$ (b) $[\frac{\pi}{2}, \pi]$ (c) (-1,1) (d) (π, ∞)

28) The number of solutions of the equation $\tan\theta = k$, $k>0$ is

(a) zero (b) only one (c) many solutions (d) two

29) The value of $\sin^{-1}(1) + \sin^{-1}(0)$ is

(a) $\frac{\pi}{2}$ (b) 0 (c) 1 (d) π

30) $\sin^{-1}(3\frac{x}{2}) + \cos^{-1}(3\frac{x}{2})$ = __________

(a) $3\frac{\pi}{2}$ (b) 6x (c) 3x (d) $\frac{\pi}{2}$

[redacted]

(a) 1 (b) $-\pi$ (c) $\frac{\pi}{2}$ (d) π

32) $\sin^{-1}x - \cos^{-1}(-x)$ = _____

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $-3\frac{\pi}{2}$ (d) $3\frac{\pi}{2}$

33) $\sec^{-1}(\frac{2}{3}) + \text{cosec}^{-1}(\frac{2}{3})$ = _____

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) π (d) $-\pi$

34) $\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3})$ = _____

(a) $\sin^{-1}(\frac{1}{\sqrt{2}})$ (b) $\sin^{-1}(\frac{1}{2})$ (c) $\tan^{-1}(\frac{1}{2})$ (d) $\tan^{-1}(\frac{1}{\sqrt{3}})$

35) The value of $\cos^{-1}(-1) + \tan^{-1}(\infty) + \sin^{-1}(1)$ = _____

(a) $-\pi$ (b) $3\frac{\pi}{2}$ (c) 30^o (d) 2π

36) The value of $\tan 135^o \cos 30^o \sin 180^o \cot 225^o$ is

(a) $1+\frac{\sqrt{3}}{2}$ (b) $1-\frac{1}{\sqrt{2}}$ (c) 1 (d) 0

37) When $A = 120^o$, $\tan A + \cot A$ = ...

(a) $-\frac{4}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $-\frac{1}{\sqrt{3}}$

38) The value of $\frac{\sin 5A - \sin 3A}{\cos 3A - \cos 5A}$

(a) cot4A (b) tan4A (c) sin4A (d) sec4A

39) The value of $\sec A \sin(270^o + A)$

(a) -1 (b) $\cos^2 A$ (c) $\sec^2 A$ (d) 1

40) If $\cos\theta = \frac{4}{5}$, then the value of $\tan\theta \sin\theta \sec\theta \operatorname{cosec}\theta \cos\theta$ is

(a) $\frac{4}{3}$ (b) $\frac{3}{4}$ (c) 1 (d) $\frac{12}{5}$

FUNCTIONS AND THEIR GRAPHS 6

The concept of function is one of the most important concepts in Calculus. It is also used frequently in every day life. For instance, the statement "Each student in the B.Tech course of Anna University will be assigned a grade at the end of the course" describes function. If we analyse this statement, we shall find the essential ingrediants of a function.

For the statement, there is a set of students, a set of possible grades, and a rule which assigns to each member of the first set a unique member of the second set. Similarly we can relate set of items in a store and set of possible prices uniquely. In Economics, it may be ncessary to link cost and output, or for that matter, profit and output.

Thus when the quantities are so related that corresponding to any value of the first quantity there is a definite value of the second, then the second quantity is called a function of the first.

6.1. FUNCTION OF A REAL VALUE

(i) Constant :

A quantity which retains the same value throughout any mathematical operation is called a *constant.* It is *conventional* to represent constants by the letters **a, b, c** etc.

For example : A *radian* is a constant angle. Any real number is a constant.

(ii) Variable:

A variable is a quantity which can have *different* values in a particular mathematical investigation. It is conventional to represent variables by the letters **x, y, z,** etc.

For example, in the equation $4x+3y = 1$, "x" and "y" are variables, for they represent the co-ordinates of any point on straight line represented by $4x+3y = 1$ and thus change their values from point to point.

There are two kinds of variables:

(i) *Independent variable* (ii) *Dependent variable*

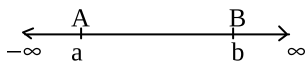
A variable is an *independent* variable when it can have any arbitrary value.

A variable is said to be a *dependent* variable when its values depend on the values assumed by some other variable.

Thus in the equation $y = 5x^2-2x+3$, "x" is the independent variable, "y" is the dependent variable and "3" is the constant. Also we can say "x" is called *Domain* and "y" is called the *Range.*

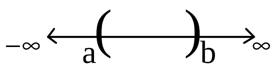
6.1.1 Intervals : Closed and Open

On the "Real line" let A and B represents two real numbers a and b respectively, with $a < b$. All points that lie between A and B are those which correspond to all real numbers x in value between a and b such that $a < x < b$. We can discuss the entire idea in the following manner.



(i) Open Interval

The set $\{x : a < x < b\}$ is called an open interval *denoted* by (a, b).

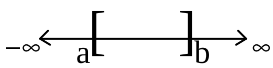


In this interval the *end points* are *not included*

For example : In the open interval (4, 6), 4 is not an element of this interval, but 5.9 is an element of this interval. 4 and 6 are not elements of (4, 6)

(ii) Closed interval

The set $\{x : a \le x \le b\}$ is called a closed interval and is denoted by [a, b].



In the interval [a, b], the end points are included.

For example : In the interval [4, 6], 4 and 6 are elements of this interval.

Also we can make a mention about *semi* closed or *semi* open intervals.

i.e. $(a, b] = \{x : a < x \le b\}$ is called left open

and $[a, b) = \{x : a \le x < b\}$ is called right open

Uniformly, in all these cases $\boxed{b-a = h}$ is called the *length* of the interval

6.1.2 Neighbourhood of a point

Let a be any real number, Let $\in > 0$ be arbitrarily small real number. Then $(a-\in, a+\in)$ is called an "$\in$" neighbourhood of the point a and denoted by $N_{a,\in}$

For example $N_{3,\frac{1}{4}} = (3-\frac{1}{4}, 3+\frac{1}{4})$

$$= \{x : \frac{11}{4} < x < \frac{13}{4}\}$$

$$N_{2,\frac{1}{5}} = (2-\frac{1}{5}, 2+\frac{1}{5})$$

$$= \{x : \frac{9}{5} < x < \frac{11}{5}\}$$

6.1.3 Functions

Definition

A function **f** from a set A to a set B is a rule which assigns to each element of A a unique element of B. The set A is called the *domain* of the function, while the set B is called the *co-domain* of the function.

Thus if **f** is a function from the set A to the set B we write $\mathbf{f} : A \to B$.

Besides **f**, we also use the notations F, g, ϕ etc. to denote functions.

If a is an element of A, then the unique element in B which **f** assigns to a is called the *value of* **f** *at* **a** or the *image of* ***a*** *under* **f** and is denoted by **f(a)**. The range is the set of all values of the function.

We can represent functions pictorially as follows :

$\mathbf{f} : A \to B$

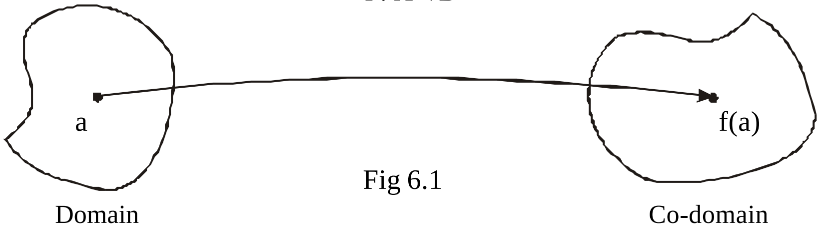


Fig 6.1

We often think of x as representing an arbitrary element of A and y as representing the corresponding value of **f** at x.

We can write y = f(x) which is read “y is a function of x” or “y is f of x” The rule of a function gives the value of the function at each element of the domain. Always the rule is a formula, but it can be other things, such as a list of *ordered pairs, a table,* or *a set of instructions.*

A function is like a machine into which you can put any number from the domain and out of which comes the corresponding value in the range.

consider, $f(x) = x^3$

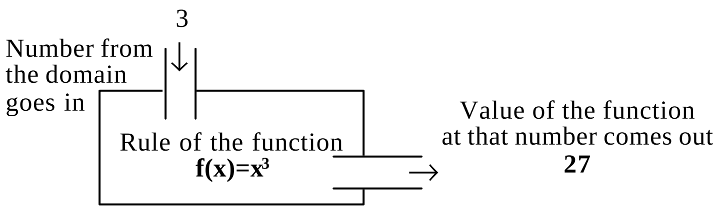


fig (6.2)

Let us consider the following equations

(i) $y = x^2-4x+3$

(ii) $y = \sin 2x$

(iii) $y = mx + c$

(iv) $V = \frac{\pi r^2 h}{3}$

(v) $s = ut + \frac{at^2}{2}$

In (i) we say that y is a function of x

In (ii) and (iii) y is a function of x. **(m and c are constants)**

In (iv) V is a function of r and h. **(two variables)**

In (v) s is a function of u, t and a. **(three variables)**

6.1.4 Tabular representation of a function

An experimental study of phenomena can result in tables that express a functional relation between the measured quantities.

For example, temperature measurements of the air at a meteorological station on a particular day yield a table.

The temperature T (in degrees) is dependent on the time t(in hours)

t	1	2	3	4	5	6	7	8	9	10
T	22	21	20	20	17	23	25	26	26.5	27.3

The table defines T as a function of t denoted by T = f(t).

Similarly, tables of trigonometric functions, tables of logarithms etc., can be viewed as functions in tabular form.

6.1.5 Graphical representation of a function.

The collection of points in the xy plane whose abscissae are the values of the independent variable and whose ordinates are the corresponding values of the function is called a *graph* of the given function.

6.1.6 The Vertical Line Test for functions

Assume that a relation has two ordered pairs with the same first coordinate, but different second coordinates. The graph of these two ordered pairs would be points on the same vertical line. This gives us a method to test whether a graph is the graph of a function.

The test :

If it is possible for a vertical line to intersect a graph at *more than one point,* then the graph is not the graph of a function.

The following graphs do not represent graph of a function:

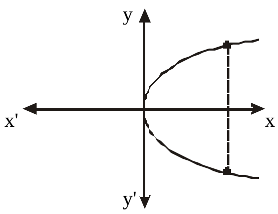


Fig 6.3

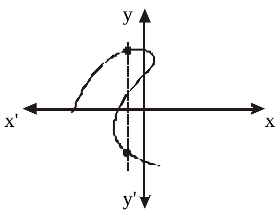


Fig 6.4

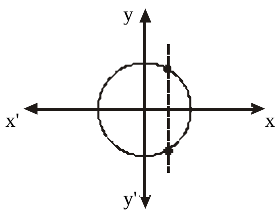


Fig 6.5

From the graphs in fig (6.3), (6.4) and (6.5) we are able to see that the vertical line meets the curves at more than one point. Hence these graphs are not the graphs of function.

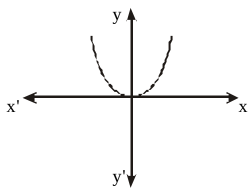


Fig 6.6

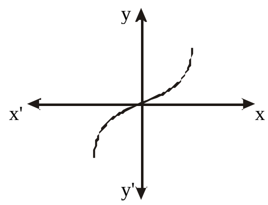


Fig 6.7

We see in fig(6.6) and (6.7) that no vertical line meet the curves at more than one point and (6.6) and (6.7) "pass" the vertical line test and hence are graphs of functions.

Example 1

(i) **What is the length of the interval $3.5 \leq x \leq 7.5$?**

(ii) **If H = {x : $3 \leq x \leq 5$} can 4.7 Î H?**

(iii) **If H = {x : $-4 \leq x < 7$} can -5 Î H?**

(iv) **Is -3 Î (-3, 0)?**

Solution:

(i) Here the interval is [a, b] = [3.5, 7.5]

Length of the interval is b-a = 7.5 - 3.5 = 4

(ii) Yes , because 4.7 is a point in between 3 and 5

(iii) No, because -5 lies outside the given interval.

(iv) In the open interval the end points are not included.

Hence $-3 \notin (-3, 0)$

Example 2

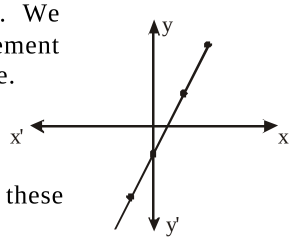
Draw the graph of the function f(x) = 3x-1

Solution:

Let us assume that y = f(x)

∴ We have to drawthe graph of y = 3x-1. We can choose any number that is possible replacement for x and then determine y. Thus we get the table.

x	0	1	2	-1	-2
y	-1	2	5	-4	-7

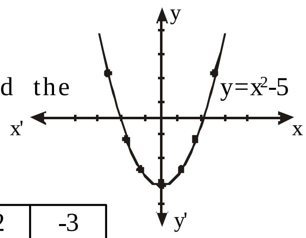


Now, we plot these poits in the xy plane these points would form a straight line.

Example 3

Draw the graph of $f(x) = x^2-5$

Solution: Let y = f(x)



We select numbers for x and find the corresponding values for y.

The table gives us the ordered pairs (0, -5), (-1, 4) and so on.

x	0	1	2	3	-1	-2	-3
y	-5	-4	-1	4	-4	-1	4

Example 4

Given the function $f(x) = x^2-x+1$

find (i) f(o) (ii) f(-1) (iii) f(x+1)

Solution:

$f(x) = x^2-x+1$

(i) $f(o) = o^2-o+1$

$= 1$

(ii) $f(-1) = (-1)^2 -(-1)+1 = 3$

(iii) $f(x+1) = (x+1)^2 - (x+1) + 1$

$= x^2+2x+1-x-1+1$

$= x^2+x+1$

Example 5

Let f : R→R defined by f(x) = $\begin{cases} \mathbf{x^2 - 4x \text{ if } x \geq 2} \\ \mathbf{x+2 \text{ if } x < 2} \end{cases}$

find i) f(-3) ii) f(5) iii) f(0)

Solution

when x = -3; f(x) = x+2 ∴ f(-3) = -3+2 = -1

when x = 5 ; $f(x) = x^2-4x$ ∴ f(5) = 25 - 20 = 5

when x= 0 ; f(x) = x+2 ∴ f(o) = 0+2 = 2

Example 6

If f(x) = sinx ; g(x) = cosx, show that : f(a+b) = f(a) g(b) + g(a) f(b) when x, a, b $\in$ R

Proof :

f(x) = sinx

$\therefore \quad f(\alpha+\beta) = \sin(\alpha+\beta)$ -------------(1)

$f(\alpha) = \sin\alpha$; $f(\beta) = \sin\beta$

$g(\alpha) = \cos\alpha$; $g(\beta) = \cos\beta$ $[\because g(x) = \cos x]$

Now,

$f(\alpha) \cdot g(\beta) + g(\alpha) \cdot f(\beta)$

$= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$

$= \sin(\alpha+\beta)$ --------------(2)

from (1) and (2), we have

$f(\alpha+\beta) = f(\alpha)\, g(\beta) + g(\alpha) \cdot f(\beta)$

Example 7

If A = {-2, -1, 0, 1, 2} and f : A® R be defined by f(x) = x^2+3 find the range of f.

Solution :

$f(x) = x^2+3$

$f(-2) = (-2)^2+3 = 4+3 = 7$

$f(-1) = (-1)^2+3 = 1+3 = 4$

$f(0) = 0 + 3 = 3$

$f(1) = 1^2 + 3 = 4$

$f(2) = 2^2 + 3 = 7$

Hence the range is the set {3, 4, 7}

Example 8

If $f(x) = \frac{1-x}{1+x}$ show that $f(-x) = \frac{1}{f(x)}$

Solution :

$$f(x) = \frac{1-x}{1+x}$$

$$\therefore \quad f(-x) = \frac{1-(-x)}{1+(-x)} = \frac{1+x}{1-x} = \frac{1}{f(x)}$$

Example 9

If f(x, y) = $ax^2 + bxy^2+cx^2y + dy^3$ find (i) f(1, 0) (ii) f(-1, 1)

Solution:

$f(x, y) = ax^2+bxy^2+cx^2y+dy^3$ ---------------(1)

To find f(1, 0) ; put x = 1 and y = 0 in (1)

$\therefore \quad f(1, 0) = a(1)^2 + 0 + 0 + 0 = a$

to find f(-1, 1) ; put x = -1 and y = 1 in (1)

$\therefore$ $f(-1, 1) = a(-1)^2 + b(-1)(1)^2 + c(-1)^2(1) + d(1)^3$

$f(-1, 1) = a - b + c + d$

Example 10

If $f(x) = x^2+3$, for $-3 \le x \le 3$, $x \in R$

(i) For which values of x, f(x) = 4?

(ii) What is the domain of f?

Solution:

(i) Given f(x) = 4

$\therefore x^2+3 = 4 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Thus for x = -1 and 1, f(x) = 4

(ii) The domain of f is $\{x : -3 \le x \le 3, x \in R\}$

Example 11

What is the domain of f for $f(x) = \frac{x-4}{x+5}$?

Solution:

Note that at x = -5 ; $f(x) = \frac{-5-4}{0} = \frac{-9}{0}$

Since we cannot divide by 0 ; x = -5 is not acceptable.

Therefore x = -5 is not in the domain of f.

Thus the domain of f is $\{x : x \in R ; x \neq -5\}$

Example 12

A group of students wish to charter a bus which holds atmost 45 people to go to an eduactional tour. The bus company requires atleast 30 people to go. It charges Rs. 100 per person if upto 40 people go. If more than 40 people go, it charges each person Rs. 100 less $\frac{1}{5}$ times the number more than 40 who go. Find the total cost as a function of the number of students who go. Also give the domain.

Solution:

Let x be the number of students who go then $30 \le x \le 45$ and x is an integer

The formula is

Total cost = (cost per student) x (number of students)

If between 30 and 40 students go , the cost per student is Rs. 100/-.

$\therefore$ The total cost is y = 100x

If between 41 and 45 students go, the cost per student is Rs. $\{100 - \frac{1}{5}(x-40)\}$

$= 108 - \frac{x}{5}$

Then the total cost is y $= (108 - \frac{x}{5})x$

$= 108x - \frac{x^2}{5}$

So the rule is $y = \begin{cases} 100x & ; 30 \le x \le 40 \\ 108x - \frac{x^2}{5} & ; 41 \le x \le 45 \end{cases}$ where x is a positive integer.

The domain is {30, 31, ..., 45}

Example 13

Find the domain and range of the function given by $f(x) = \log_{10}(1+x)$

Solution:

We know, log of a negative number is not defined over R and log0 = $-\infty$

$\therefore$ $\log_{10}(1+x)$ is not real valued for $1+x < 0$ or for $x<-1$ and log (1+x) tends to $-\infty$ as $x \to -1$

Hence the domain of f is $(-1, \infty)$

i.e. all real values greater than -1. The range of this function is R^+ (set of all positive real numbers)

Example 14

Find the domain of the function $f(x) = \sqrt{x^2 - 7x + 12}$

Solution :

$f(x) = \sqrt{(x-3)(x-4)}$

f(x) is a real valued function only when (x-3) (x-4) > 0

ie when x lies outside '3' and '4'

$\therefore$ The domain of f(x) is x > 4 and x < 3 i.e. $[-\infty, 3)$ and $(4, \infty]$

EXERCISE 6.1

1) Draw the graph of the line y = 3

2) If f(x) = tanx and f(y) = tany, prove that $f(x-y) = \frac{f(x) - f(y)}{1 + f(x) f(y)}$

3) If $f(x) = \frac{x+\tan x}{x+\sin x}$, prove that $f\left(\frac{\pi}{4}\right) = \frac{\pi+4}{\pi+2\sqrt{2}}$

4) If $f(x) = \frac{1+x^2+x^4}{x^2}$ prove that $f\left(\frac{1}{x}\right) = f(x)$

5) If $f(x) = x^2-3x+7$, find $\frac{f(x+h)-f(x)}{h}$

6) If $f(x) = \sin x + \cos x$, find $f(0) + f\left(\frac{\pi}{2}\right) + f(\pi) + f\left(3\frac{\pi}{2}\right)$

7) Find the domain of $g(x) = \sqrt{1-\frac{1}{x}}$

8) A travel agency offers a tour. It charges Rs. 100/- per person if fewer than 25 people go. If 25 people or more, upto a maximum of 110, take the tour, they charge each person Rs. 110 less $\frac{1}{5}$ times the number of people who go. Find the formulae which express the total charge C as a function in terms of number of people n who go . Include the domain of each formulae.

9) Find the domain of the function $f(x) = \sqrt{x^2 - 5x + 6}$

10) Which of the follwing graphs do not represent graph of a function?

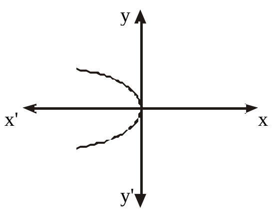


Fig (i)

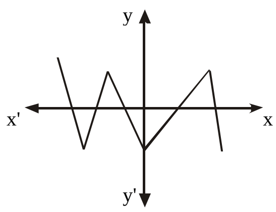


Fig (ii)

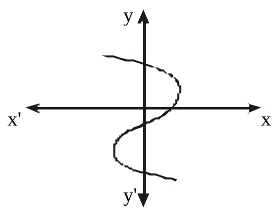


Fig (iii)

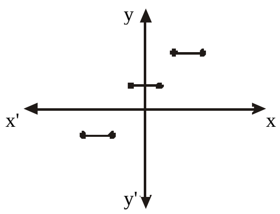


Fig (iv)

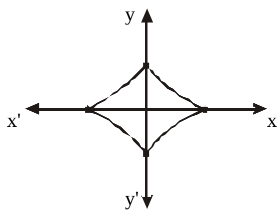


Fig (v)

11) If $f(x) = \sin x$; $g(x) = \cos x$,
show that : $f(\alpha-\beta) = f(\alpha)\ g(\beta) - g(\alpha) . f(\beta)$; $\alpha, \beta, x \in R$

12) For $f(x) = \frac{x-1}{3x+5}$; write the expressions $f(\frac{1}{x})$ and $\frac{1}{f(x)}$

13) For $f(x) = \sqrt{x^2+4}$, write the expression f(2x) and f(0)

14) Draw the graph of the funtion $f(x) = 5x-6$

15) Draw the graphs of the functions $f(x) = x^2$ and $g(x) = 2x^2$

16) If $f(x) = x^2-4$, Draw the graphs of f(x), 2f(x), and -f(x) in the same plane.

6.2 CONSTANT FUNCTION AND LINEAR FUNCTION

6.2.1. Constant function

A function whose range consists of just one element is called a *constant function* and is written as f(x) = a constant for every $x \in$ domain set.

For example : f(x) = 2 and f(x) = -3 are constant functions.

The figure 6.8 represents the constant function

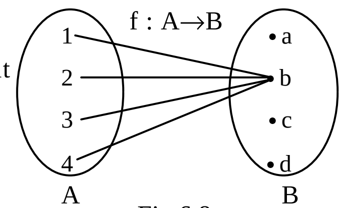


Fig 6.8

We can draw the graph of the constant function f(x) = c, where c is a constant.

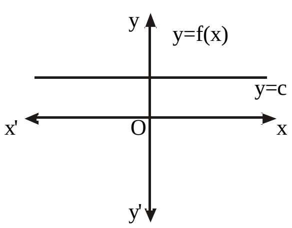


Fig 6.9

We can easily observe that in fig (6.9); the graph of the constant function represents a straight line parellel to x-axis.

Observation :

The relation set H = [(1, 5), (2, 5), (3, 5), (4, 5)} is a constant function.

6.2.2 Linear function

A *Linear function* is a function whose rule is of the form f(x) = ax+b, where a and b are real numbers with $a \neq 0$.

We shall see that the graph of a linear function is a straight line.

6.2.3 Slope of the line *l*

If *l* is a line which is not vertical and if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points on the line, then the slope of the line usually denoted by **m** is given by

$$\mathbf{m} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference in y coordinate}}{\text{Difference in x coordinate}}$$

$\therefore$ the linear function f(x) = ax+b, $(a \neq 0)$ may be written as f(x) = mx+c, where m is the slope of the line ; and c is the y intercept.

Observation:

(i) If the slope of the line m is positive, then the line goes upward as it goes to the right.

(ii) If m is negative then the line goes downward as it goes to the right

(iii) If m = 0 the line is horizontal

(iv) If m is undefined the line is vertical.

6.2.4 A linear function denotes the equation of a straight line which can be expressed in the following different forms

(i) $y = mx+c$, **(slope - intercept form)**

(ii) $y-y_1 = m(x-x_1)$: **(slope-point form)**

(iii) $\frac{x}{a} + \frac{y}{b} = 1$; **(intercept form)**

(iv) $\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$; **(two point form)**

Variables of these functions have no powers more than one. The equations describing the relationship are called *first - degree* equations or linear equation.

6.2.5 Application of linear functions

(i) Salary of an employee can be expressed as a linear function of time.

(ii) The life expectancy of a particular sex may be expressed through linear function of year (t)

(iii) Linear relationship between price and quantity.

Example 15

The salary of an exmployee in the year 2002 was Rs. 7,500. In 2004, it will be Rs. 7750. Express salary as a linear function of time and estimate his salary in the year 2005.

Solution:

Let S represent Salary (in Rs.) and t represent the year (t)

year	Salary (Rs.)
2002 (t_1)	7,500 (S_1)
2004 (t_2)	7,750 (S_2)
2005 (t)	? (S)

The equation of the straight line representing salary as a linear function of time is

$$S - S_1 = \frac{S_2 - S_1}{t_2 - t_1}(t - t_1)$$

$$S - 7500 = \frac{7750 - 7500}{2004 - 2002}(t - 2002)$$

$$S - 7500 = \frac{250}{2}(t - 2002)$$

$$S = 7,500 + 125\,(t - 2002)$$

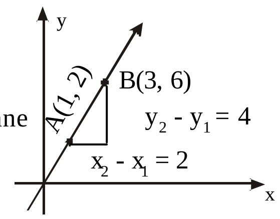
when t = 2005

$$S = 7500 + 125\,(2005 - 2002)$$
$$= 7500 + 125\,(3)$$
$$= 7500 + 375$$
$$= 7875$$

The estimated salary in the year 2005 is Rs. 7,875.

Example 16

Find the slope of straight line containing the points (1, 2) and (3, 6)



Solution:

Plot the points (1, 2) and (3, 6) in the xy plane and join them.

Slope $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{3-1} = 2$

6.3. POWER FUNCTION

6.3.1 Power function

A function of the form $f(x) = ax^n$, where a and n are non-zero constants is called a *power function*.

For example $f(x) = x^4$, $f(x) = \frac{1}{x^2}$ and $f(x) = 3x^{\frac{1}{2}}$ etc. are power function.

6.3.2 Exponential function

If $a > 0$, the exponential function with base a is the function 'f' defined by

$$f(x) = a^x \text{ where x is any real number.}$$

For different values of the base a, the exponential function $f(x) = a^x$ (and its graph) have different characteristics as described below:

6.3.3 Graph of $f(x) = a^x$, where $a>1$

Study of Graph 2^x

In $f(x) = a^x$, let $a = 2$ $\therefore$ $f(x) = 2^x$

For different values of x. The corresponding values of 2^x are obtained as follows:

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

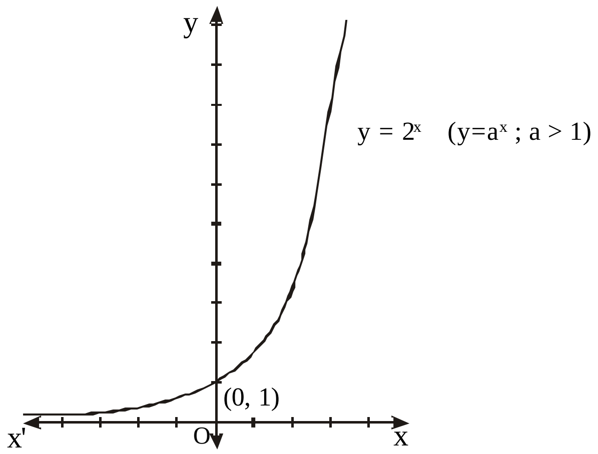


Fig 6.10

Observation:

(i) Graph of 2^x is strictly increasing. To the left of the graph, the x axis is an *horizontal asymptote*

(ii) The graph comes down closer and closer to the negative side of the x-axis.

(iii) Exponential functions describe situations where growth is taking place.

6.3.4 Graph of $f(x) = a^x$, when $a < 1$

Study of Graph $\left(\frac{1}{2}\right)^x$

$f(x) = a^x$; Let $a = \frac{1}{2}$ $\therefore f(x) = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
$\left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

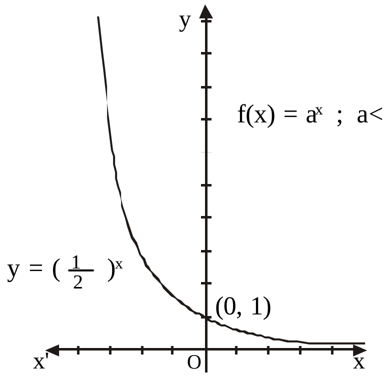


Fig 6.11

Observation :

(i) The curve is strictly decreasing

(ii) The graph comes down closer and closer to the positive side of the x-axis

(iii) For different values of a, the graphs of $f(x) = a^x$ differ in *steepness*

(iv) If $a > 1$, then $0 < \frac{1}{a} < 1$, and the two graphs $y = a^x$ and $y = \left(\frac{1}{a}\right)^x$ are reflections of each other through the y axis

(v) If a = 1, the graph of $f(x) = a^x$ is a horizontal straight line

(vi) The domain and the range of $f(x) = a^x$ is given by $R \rightarrow (0, \infty)$

6.3.5 Graph of $f(x) = e^x$

The most used power function is $y = e^x$, where e is an irrational number whose value lies between 2 and 3. (e = 2.718 approxi). So the graph of e^x is similar to the graph of $y = 2^x$.

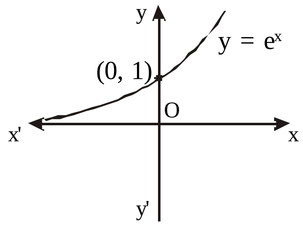


Fig 6.12

6.3.6 Logarithmic Functions

If $0 < a < 1$ or $a > 1$, then $\log_a x = y$ if and only if $a^y = x$

The function $f(x) = \log_a x$ is not defined for all values of x. Since a is positive, a^y is positive,. Thus with $x = a^y$, we see that, if $0<a<1$ or $a>1$; $\log_a x$ is defined only for $x > 0$.

If $0 < a < 1$ or $a > 1$, then (i) $\log_a a = 1$ and (ii) $\log_a 1 = 0$

In the fig. 6.13 the graph of $f(x) = \log_a x$ is shown. This graph is strictly increasing if $a>1$ and strictly decreasing if $0<a<1$.

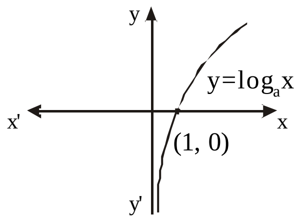


Fig 6.13

Observation :

(i) Since $\log_a 1 = 0$, the graph of $y = \log_a x$ crosses the x axis at x = 1

(ii) The graph comes down closer and closer to the negative side of y-axis

(iii) For different values of a, the graphs of $y = \log_a x$ differ in steepness

(iv) The domain and the range of $y = \log_a x$ is given by $(0, \infty) \rightarrow R$

(v) The graphs of $f(x) = a^x$ and $g(x) = \log_a x$ are symmetric about the line y = x

(vi) By the principle of symmetry the graph of $\mathbf{\log_e x}$ can be obtained by reflecting the graph of $\mathbf{e^x}$ about the line y = x, which is shown clearly in the following diagram.

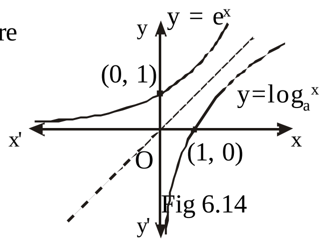


Fig 6.14

6.4 CIRCULAR FUNCTIONS

6.4.1 Periodic Functions

If a variable angle θ is changed to θ+α, α being the least positive constant, the value of the function of θ remains unchanged, the function is said to be *periodic* and α is called the *period* of the function.

Since, $\sin(\theta+2\pi) = \sin\theta$, $\cos(\theta+2\pi) = \cos\theta$. We say sin θ and cos θ are functions each with period 2π. Also we see that $\tan(\theta+\pi) = \tan\theta$ hence we say that tan θ is period with π.

Now, we need only to find the graphs of sine and cosine functions on an interval of length 2π, say $0 \le \theta \le 2\pi$ or $-\frac{\pi}{2} \le \theta \le 3\frac{\pi}{2}$ and then use $f(\theta+2\pi) = f(\theta)$ to get the graph everywhere. In determining their graphs the presentation is simplified if we view these functions as *circular functions*.

We first consider the sine function, Let us see what happens to sinx as x increases from 0 to 2π.

6.4.2 Graph of sinx. Consider sine function in 0≤x≤2p

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
sinx	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$

The graph of sinx is drawn as below:

y = sinx

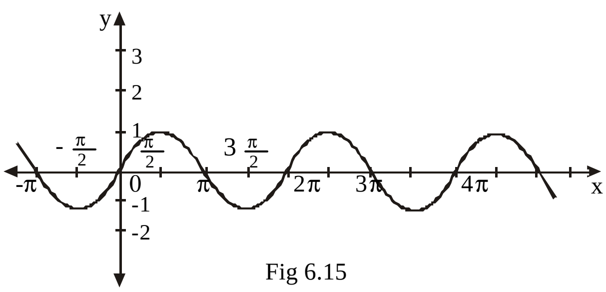


Fig 6.15

Observation:

(i) The scale on x-axis is different from the scale on the y axis in order to show more of the graph.

(ii) The graph of sinx has no break anywhere i.e. it is continuous.

(iii) It is clear from the grpah that maximum value of sinx is 1 and the minimum value is -1. ie the graph lies entirely between the lines y = 1 and y =-1

(iv) Every value is repeated after an interval of 2π. i.e. the function is periodic with 2π.

6.4.3 Graph of f(x) = cosx

Consider the cosine function. We again use the interval $0 \le x \le 2\pi$

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$3\frac{\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
cosx	-1	0	1	0	-1	0	1	0	-1

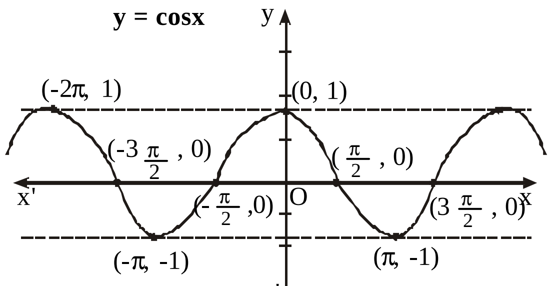


Fig 6.16

Observation:

(i) The graph of cosx has no break anywhere i.e. it is continuous

(ii) It is clear from the graph that the maximum value of cosx is 1 and minimum value is -1 ie. the graph lies entirely between the lines y = 1 andy =-1

(iii) The graph is symmetrical about the y-axis

(iv) The function is periodic with period 2π.

6.4.4 Graph of tanx

Since division by 0 is undefined tan $\frac{\pi}{2}$ is meaningless. In tanx, the variable represents any real number. Note that the function value is 0 when x = 0 and the values increase as x increases toward $\frac{\pi}{2}$.

As we approach $\frac{\pi}{2}$, the tangent values become very large. Indeed, they increase without bound. The dashed vertical lines are not part of the graph. They are *asymptotes*. The graph approaches each asymptote, but never reaches it because there are no values of the function for $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc.

y = tanx

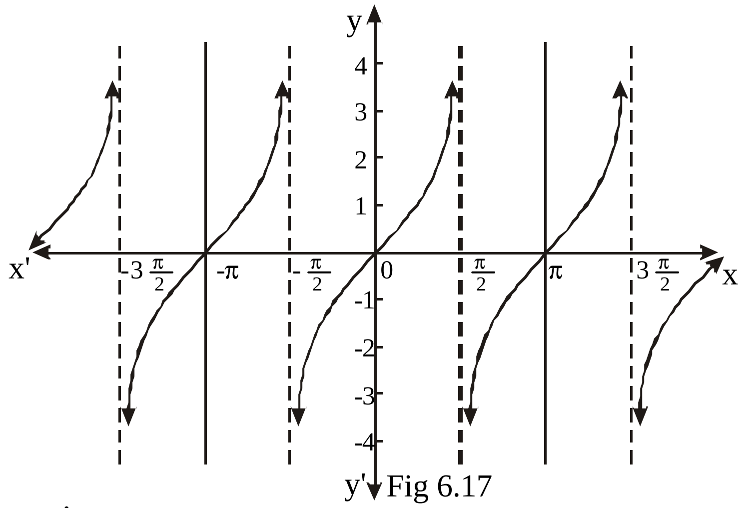


Fig 6.17

Observation :

(i) The graph of tan x is discontinuous at points when

$x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, ...$

(ii) tanx may have any numerical value positive or negative

(iii) tanx is a periodic function with period π.

Example 17

Is the tangent function periodic? If so, what is its period? What is its domain and range?

Solution :

From the graph of y = tanx(fig 6.17), we see that the graph from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ repeats in the interval from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ consequently, the tangent function is periodic, with a period π

Domain is {x ; x ≠ $\frac{\pi}{2}$ + kπ, k is an integer}

Range is R (set of all real numbers)

Example 18

What is the domain of the secant function?

Solution :

The secant and cosine functions are reciprocals. The secant function is undefined for those numbers for which cosx = 0. The domain of the secant function is the set of all real numbers except $\frac{\pi}{2}$ + kπ, k is an integer.

ie. {x : x ≠ $\frac{\pi}{2}$ + kπ, k is an integer}

Example 19

What is the period of this function?

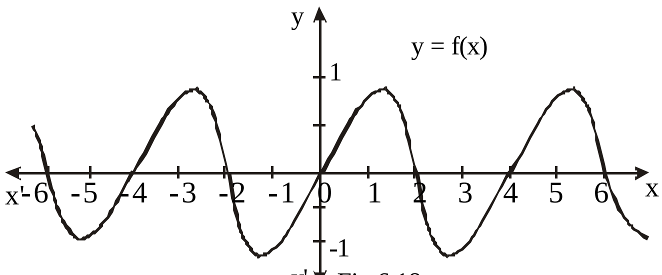


Fig 6.18

Solution:

In the graph of the function f, the function values repeat every four units. Hence f(x) = f(x+4) for any x, and if the graph is translated four units to the left or right, it will coincide with itself. Therefore the period of this function is 4.

6.5 ARITHMETIC OF FUNCTION

6.5.1 Algebraic functions

Those functions which consist of a finite number of terms involving powers, and roots of independent variable and the four fundamental operations of addition, subtraction, multiplication and division are called algebraic functions.

For example, $\sqrt{3x+5}$, $\sqrt[7]{x}$, $4x^2-7x+3, 3x-2, 2x^3$ etc are algebraic functions

Also, algebraic functions include the rational integral function or polynomial

$$f(x) = a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....... +a_n$$

where a_0, a_1, a_2, ..., a_n are constants called *coefficients* and n is non-negative integer called degree of the polynomial. It is obvious that this function is defined for all values of x.

6.5.2 Arithmetic operations in the set of functions

Consider the set of all real valued functions having the same domain D. Let us denote this set of functions by E.

Let f, g $\in$ E. ie., functions from D into R.

The *arithmetic of functions*; $f \pm g$, fg and $f \div g$ are defined as follows:

$(f + g)(x) = f(x) + g(x), \forall\, x \in D$

$(f - g)(x) = f(x) - g(x),$

$(fg)(x) = f(x)\, g(x),$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}; \; g(x) \neq 0$

Observation :

(i) The domain of each of the functions f + g, f - g, fg is the same as the common domain D of f and g.

(ii) The domain of the quotient $\frac{f}{g}$ is the common domain D of the two functions f and g excluding the numbers x for which g(x) = 0

(iii) The product of a function with itself is denoted by f^2 and in general product of f taken 'n' times is denoted by f^n where n is a natural number.

6.5.3 Computing the sum of functions

(i) For example consider f(x) = 3x+4 ; g(x) =5x-2 be the two linear functions then their sum (f+g) (x) is

f(x) = 3x+4

g(x) = 5x-2

(+) -------   ------- (+)

f(x)+g(x) = (3x+5x) + (4-2)

$\therefore$ f(x)+g(x) = 8x+2 = (f+g) (x)

(ii) Consider $f(x) = 3x^2-4x+7$ and $g(x) = x^2-x+1$ be two quadratic functions then the sum of f(x) and g(x) is f(x)+g(x) $= (3x^2-4x+7) + (x^2-x+1)$

$= (3x^2+x^2) + (-4x-x) + (7+1)$

f(x) + g(x) $= 4x^2-5x+8 = (f+g)(x)$

(iii) Consider $f(x) = \log_e x$; $g(x) = \log_e(5x)$ be two logarithmic functions then the sum (f+g) (x) is $f(x)+g(x) = \log_e x + \log_e 5x = \log_e 5x^2$. Observe that here $f(x) + f(y) \neq f(x+y)$

(iv) Consider, $f(x) = e^x$ and $f(y) = e^y$ be two exponential functions, then the sum f(x)+f(y) is e^x+e^y

(v) Consider, f(x) = sinx, g(x) = tanx then the sum f(x)+g(x) is sinx + tanx

6.5.4 Computing Difference of functions

(i) Consider $f(x) = 4x^2-3x+1$ and $g(x) = 2x^2+x+5$ then (f-g) (x) = f(x)-g(x) is $(4x^2-2x^2) + (-3x-x) + (1-5) = 2x^2-4x-4$

(ii) Consider $f(x) = e^{3x}$ and $g(x) = e^{2x}$ then

(f-g) (x) = f(x) - g(x)

$= e^{3x} - e^{2x}$

(iii) Consider $f(x) - \log_e{}^{5x}$ and $g(x) = \log_e{}^{3x}$ then

(f-g) (x) is f(x) - g(x) $= \log_e{}^{5x} - \log_e{}^{3x}$

$= \log_e\left(\frac{5x}{3x}\right) = \log_e \frac{5}{3}$

6.5.5 Computing the Product of functions

(i) Consider f(x) = x+1 and g(x) = x-1 then the product f(x) g(x) is (x+1) (x-1) which is equal to x^2-1

(ii) Consider, $f(x) = (x^2-x+1)$ and g(x) = x+1 then the product f(x) g(x) is

$(x^2-x+1)(x+1) = x^3-x^2+x+x^2-x+1$

$= x^3+1$

(iii) Consider, $f(x) = \log_a x$ and $g(x) = \log_a 3x$ then $(fg)x = f(x)g(x) = \log_a x \ \log_a 3x$

(iv) Consider, $f(x) = e^{3x}$; $g(x) = e^{5x}$ then the product f(x) g(x) is $e^{3x} . e^{5x} = e^{3x+5x} = e^{8x}$

6.5.6 Computing the Quotient of functions

(i) Consider $f(x) = e^{4x}$ and $g(x) = e^{3x}$

then $\frac{f(x)}{g(x)}$ is $\frac{e^{4x}}{e^{3x}} = e^{4x-3x} = e^x$

(ii) Consider, $f(x) = x^2-5x+6$; $g(x) = x-2$ then the quotient

$\frac{f(x)}{g(x)}$ is $\frac{x^2-5x+6}{(x-2)}$

which is equal to $\frac{(x-3)(x-2)}{x-2} = x-3$

Example 20

Given that $f(x) = x^3$ and $g(x) = 2x+1$

Compute (i) (f+g) (1) (ii) (f-g) (3) (iii) (fg) (0) (iv) (f ÷ g) (2)

Solution:

(i) We know $(f+g)(x) = f(x) + g(x)$

$\therefore$ $(f+g)(1) = f(1) + g(1)$

$= (1)^3 + 2(1)+1 = 4$

(ii) We know $(f-g)(x) = f(x) - g(x)$

$\therefore$ $(f-g)(3) = f(3) - g(3)$

$= (3)^3 - 2(3) - 1 = 20$

(iii) We know $(fg)(x) = f(x)\, g(x)$

$\therefore$ $fg(0) = f(0)\, g(0)$

$= (0^3)(2 \times 0 + 1) = 0$

(iv) We know $(f \div g)(x) = f(x) \div g(x)$

$\therefore$ $(f \div g)(2) = f(2) \div g(2)$

$= 2^3 \div [2(2) + 1]$

$= 2^3 \div 5 = \frac{8}{5}$

6.6 SOME SPECIAL FUNCTIONS

6.6.1 Absolute value function f(x) = |x|

Finding the absolute value of a number can also be thought of in terms of a function, the absolute value function $f(x) = |x|$. The domain of the absolute value function is the set of real numbers ; the range is the set of positive real numbers

The graph has two parts,

For $x \geq 0$, $f(x) = x$

For $x < 0$, $f(x) = -x$

f(x) = |x|

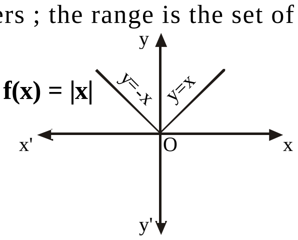


Fig (6.19)

Observation :

(i) The graph is symmetrical about the y-axis

(ii) At x = 0, |x| has a minimum value, 0

6.6.2 Signum function

The signum function is defined as

$$y = f(x) = \begin{cases} \frac{|x|}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

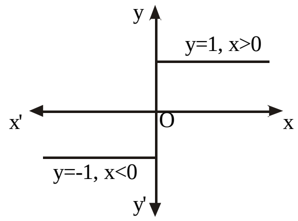


Fig (6.20)

or $$f(x) = \begin{cases} \frac{x}{x} = 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -\frac{x}{x} = -1 & \text{for } x < 0 \end{cases}$$

For x > 0, the graph of y = 1 is a straight line parallel to x-axis at a unit distance above it. In this graph, the point corresponding to x = 0 is excluded for x = 0, y = 0, we get the point (0, 0) and for x < 0, the graph y =-1 is a straight line parallel to x-axis at a unit distance below it. In this graph, the point corresponding to x= 0 is excluded.

6.6.3 Step function

The greatest integer function,

f(x) = [x], is the greatest integer that is less than or equal to x.

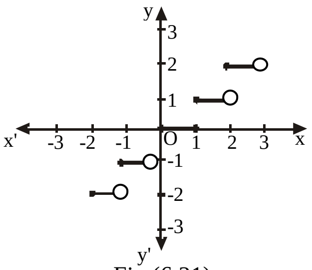


Fig (6.21)

In general,

For $0 \leq x < 1$, we have f(x) = [x] = 0

$1 \leq x < 2$, we have f(x) = [x] = 1

$2 \leq x < 3$, we have f(x) = [x] = 2

$-2 \leq x < -1$, we have f(x) = [x] = -2

$-5 \leq x < -4$, we have f(x) = [x] = -5 and so on.

In particular, [4.5] = 4, [-1] = -1, [-3.9] = -4

We can use the pattern above to graph f(x) for x between any two integers, and thus graph the function for all real numbers.

6.7 INVERSE OF A FUNCTION

6.7.1 One-one function

If a function relates any two distinct elements of its domain to two distinct elements of its co-domain, it is called a one-one function. f: A→B shown in fig.6.22 is one-one function.

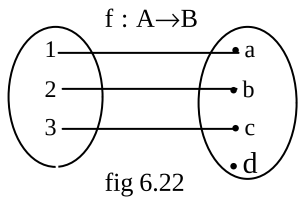


fig 6.22

6.7.2 On-to function

For an 'onto' function f: A→B, range is equal to B.

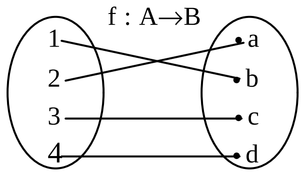


fig 6.23

6.7.3 Inverse function

Let f: A→B be a one-one onto mapping, then the maping $f^{-1}: B \to A$ which associates to each element b∈B the element a∈A, such that f(a) = b is called the inverse mapping of the mapping f: A→B.

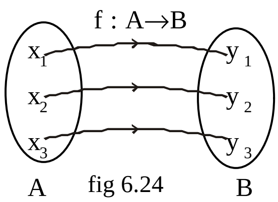


fig 6.24

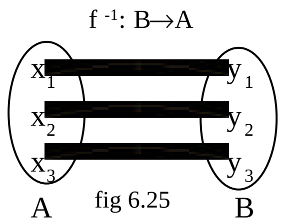


fig 6.25

from fig (6.24) $f(x_1) = y_1$ etc.   from fig (6.25) $f^{-1}(y_1) = x_1$ etc.

Observation :

(i) If f : A →B is one-one onto, then $f^{-1}: B \to A$ is also one-one and onto

(ii) If f: A →B be one-one and onto, then the inverse mapping of f is unique.

(iii) The domain of a function f is the range of $\mathbf{f^{-1}}$ and the range of **f** is the domain of $\mathbf{f^{-1}}$.

(iv) If **f** is continuous then $\mathbf{f^{-1}}$ is also continuous.

(v) Interchanging first and second numbers in each ordered pair of a relation has the effect of interchanging the x-axis and the y-axis. Interchanging the x-axis and the y-axis has the effect of reflecting the graph of these points across the diagonal line whose equation is y = x.

Example 21

Given f(x) = 2x+1, find an equation for $f^{-1}(x)$.

Let y = 2x+1, interchange x and y

$\therefore$ x = 2y+1 => $y = \frac{x-1}{2}$

Thus $f^{-1}(x) = \frac{x-1}{2}$

6.7.4 Inverse Trignometric functions

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ are the inverses of sinx, cosx and tanx respectively.

$\sin^{-1}x$:Suppose $-1 \leq x \leq 1$. Then $y = \sin^{-1}x$ if and only if x = siny and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\cos^{-1}x$: Suppose $-1 \leq x \leq 1$. Then $y = \cos^{-1}x$ if and only if x = cosy and $0 \leq y \leq \pi$

$\tan^{-1}x$: Suppose x is any real number. Then $y = \tan^{-1}x$ if and only if x = tany and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

[redacted]

and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$

$\sec^{-1}x$: Suppose $|x| \geq 1$, then $y = \sec^{-1}x$ if and only if and only if x = secy and $0 \leq y \leq \pi$ $y \neq \frac{\pi}{2}$

[redacted]

if x = coty and $0 < y < \pi$

Two points symmetric with respect to a line are called *reflections* of each other across the line. The line is known as a *line of symmetry*.

(i) From the fig 6.26 we see that the graph of $y = \sin^{-1}x$ is the reflection of the graph of y = sinx across the line y = x

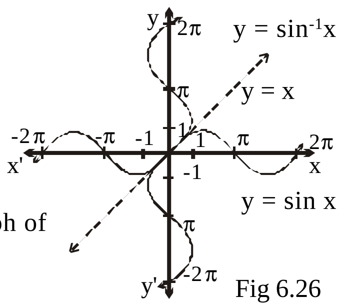


Fig 6.26

(ii) The graphs of $y = \cos x$ and $y = \cos^{-1}x$ are given in fig. 6.27 and 6.28 respectively

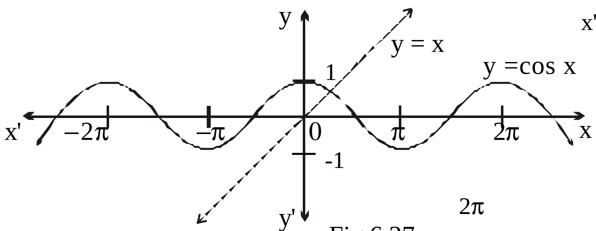


Fig 6.27

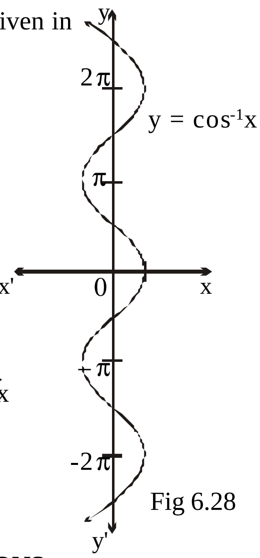


Fig 6.28

6.8 MISCELLANEOUS FUNCTIONS

6.8.1 Odd Function

A function f(x)is said to be *odd function* if $f(-x) = -f(x)$, for all x

eg : 1. $f(x) = \sin x$ is an odd function

consider ; $f(-x) = \sin(-x) = -\sin x = -f(x)$

2. $f(x) = x^3$ is an odd function ;

consider, $f(-x) = (-x)^3 = -x^3 = -f(x)$

6.8.2 Even function

A function f(x) is said to be *even function* if $f(-x) = f(x)$, for all x

eg. 1 $f(x) = \cos x$ is an even function

consider, $f(-x) = \cos(-x) = \cos x = f(x)$

2. $f(x) = x^2$ is an even function

consider, $f(-x) = (-x)^2 = x^2 = f(x)$

Observation :

(i) If f(x) is an even function then the graph of f(x) is symmetrical about y axis

(ii) There is always a possibility of a function being neither even nor odd.

(iii) If f(x) is an odd function then the graph of f(x) is symmetrical about origin.

6.8.3 Composite Function (Function of a function)

Let $f : A \to B$ and $g : B \to C$ be two functions then the function $gof : A \to C$ defined by

$(gof)(x) = g[f(x)]$, for all $x \in A$ is called composition of the two functions f and g

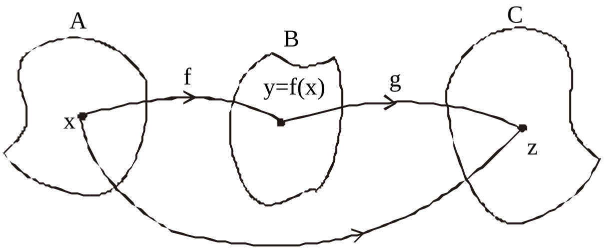


Fig 6.29

i.e., we have $z = g(y) = g[f(x)]$

Observation :

(i) In the operation (gof) we operate first by f and then by g.

(ii) $fog \neq gof$ in general

(iii) $fo(goh) = (fog)oh$ is always true

(iv) $(fof^{-1})(x) = x$, where f^{-1} is inverse of 'f'

(v) gof is onto if f and g are separately onto.

Example 22

Prove that $f(x) = |x|$ is even

Proof:

$f(x) = |x|$

$\therefore$ $f(-x) = |-x| = |x| = f(x)$

$\Rightarrow f(-x) = f(x)$

Hence $f(x) = |x|$ is even

Example 23

Prove that $f(x) = |x-4|$ is neither even nor odd.

Proof :

$f(x) = |x-4|$ $\therefore$ $f(-x) = |-x-4|$

$\therefore = |-(x+4)|$

$= |x+4|$

$\therefore$ $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

$\therefore$ $f(x) = |x-4|$ is neither even nor odd

Example 24

Prove that $f(x) = e^{x}-e^{-x}$ is an odd function

Proof:

$f(x) = e^{x}-e^{-x}$

$f(-x) = e^{-x}-e^{-(-x)}$

$= e^{-x}-e^{x} = -[e^{x}-e^{-x}]$

$= - f(x)$

Hence $f(x) = e^{x}-e^{-x}$ is an odd function

Example 25

Let $f(x) = 1-x$; $g(x) = x^{2}+2x$ both f and g are from R® R verify that fog ¹ gof

Solution:

L.H.S. $(fog)x = f(x^{2}+2x)$

$= 1-(x^{2}+2x)$

$= 1-2x-x^{2}$

R.H.S. $(gof)x = g(1-x)$

$= (1-x)^{2} + 2(1-x)$

$= 3-4x+x^{2}$

L.H.S. $\neq$ R.H.S.

Hence $fog \neq gof$

Example 26

Let $f(x) = 1-x$, $g(x) = x^{2}+2x$ and $h(x) = x+5$. Find (fog) oh

Solution:

$g(x) = x^{2}+2x \therefore (fog)\,x = f[g(x)]$

$= f(x^{2}+2x)$

$= 1-2x-x^{2}$

$\{(fog)\,oh\}\,(x) = (fog)\,(x+5)$

$= 1-2\,(x+5)-(x+5)^{2}$

$= -34-12x-x^{2}$

Example 27

Suppose $f(x) = |\,x\,|$, $g(x) = 2x$ Find (i) $f\{g(-5)\}$ (ii) $g\{f(-6)\}$

Solution:

(i) $f\{g(-5)\}$

$g(x) = 2x \therefore g(-5) = 2x(-5) = -10$

$f((g(-5)) = f(-10) = |-10| = 10$

(ii) g{f(-6)}

f(x) = | x |

∴ f(-6) = |-6| = 6

g{f(-6)} = g(6) = 2 x 6 = 12

Example 28

f(x) = 2x+7 and g(x) = 3x+b find "b" such that f{g(x)} = g{f(x)}

L.H.S. f{g(x)}

f{g(x)} = f{3x+b}

= 2(3x+b) + 7

= 6x+2b+7

R.H.S. g{f(x)}

g{f(x)}= g{2x+7}

= 3(2x+7) + b

= 6x+21+b

since f{g(x)} = g{f(x)}

we have 6x+(2b+7) = 6x+(b+21)

∴ 2b+7 = b+21

b = 21-7

b = 14

EXERCISE 6.2

1) Prove that (i) $f(x) = x^2 + 12x + 36$ is neither even nor odd function

(ii) $f(x) = 2x^3 + 3x$ is an odd function

2) If f(x) = tanx, verify that

$$f(2x) = \frac{2f(x)}{1-\{f(x)\}^2}$$

3) If $\phi(x) = \log \frac{1-x}{1+x}$ verify that $\phi(a) + \phi(b) = \phi\left(\frac{a+b}{1+ab}\right)$

4) If f(x) = logx ; $g(x) = x^3$, write the expressions for

a) f{g(2)} b) g{f(2)}

5) If $f(x) = x^3$ and g(x) = 2x+1 find the following

(i) (f+g) (0) (ii) (f+g) (-2) (iii) (f-g) (-2)

(iv) (f-g) ($\sqrt{2}$) (v) f(g) (1- $\sqrt{2}$) (vi) (fg) (0.5)

(vii) (f ÷ g) (0) (viii) (f÷ g) (-2) also find the domain of f ÷ g

6) Given f(x) = sinx, g(x) = cosx compute

(i) (f+g) (0) and (f+g) ($\frac{\pi}{2}$)

(ii) (f-g) $(-\frac{\pi}{2})$ and (f-g) (π)

(iii) (fg) $(\frac{\pi}{4})$ and (fg) $(-\frac{\pi}{4})$

(iv) (f÷g) (0) and (f÷g) (π) ; Also find the domain of $(\frac{f}{g})$

7) Obtain the domains of the following functions

(i) $\frac{1}{1+\cos x}$ (ii) $\frac{x}{1-\cos x}$ (iii) $\frac{1}{\sin^2 x-\cos^2 x}$

(iv) $\frac{|x|}{|x|+1}$ (v) $\frac{1+\cos x}{1-\cos x}$ (vi) tanx

8) The salary of an employee in the year 1975 was Rs. 1,200. In 1977 it was Rs. 1,350. Express salary as a linear function of time and calculate his salary in 1978.

9) The life expectancy of females in 2003 in a country is 70 years. In 1978 it was 60 years. Assuming the life expectancy to be a linear function of time, make a prediction of the life expectancy of females in that country in the year 2013.

10) For a linear function f, f(-1) = 3 and f(2) = 4
(i) Find an equation of f
(ii) Find f(3) (iii) Find a such that f(a) = 100

EXERCISE 6.3

Choose the correct answer

1) The point in the interval (3, 5] is
(a) 3 (b) 5.3 (c) 0 (d) 4.35

2) Zero is not a point in the interval
(a) $(-\infty, \infty)$ (b) $-3\leq x\leq 5$ (c) $-1<x\leq 1$ (d) $[-\infty, -1]$

3) Which one of the following functions has the property $f(x) = f(\frac{1}{x})$

(a) $f(x) = \frac{x^2+1}{x}$ (b) $f(x) = \frac{x^2-1}{x}$ (c) $f(x) = \frac{1-x^2}{x}$ (d) f(x) = x

4) For what value of x the function $f(x) = \sqrt{\frac{x}{2}}$ is not real valued?

(a) $x < 0$ (b) $x \leq 0$ (c) $x < 2$ (d) $x \leq 2$

5) The domain of the function $f(x) = \frac{x-4}{x+3}$ is

(a) $\{x / x \neq -3\}$ (b) $\{x / x \geq -3\}$ (c) { } (d) R

6) The period of the function f(x) = sinx is 2π, therefore what is the period of the function g(x) = 3sinx?

(a) 3π (b) 6π (c) 2π (d) $\frac{\pi}{3}$

7) The period of the cotangent function is

(a) 2π (b) π (c) 4π (d) $\frac{\pi}{2}$

8) The reciprocals of sine and cosine functions are periodic of period

(a) π (b) $\frac{1}{2\pi}$ (c) 2π (d) $\frac{2}{\pi}$

9) If f(x) = -2x+4 then $f^{-1}(x)$ is

(a) 2x-4 (b) $-\frac{x}{2} + 2$ (c) $-\frac{1}{2}x+4$ (d) 4-2x

10) If f(x) = $\log_5 x$ and g(x) = $\log_x 5$ then (fg) (x) is

(a) $\log_{25} x^2$ (b) $\log_{x^2} 25$ (c) 1 (d) 0

11) If f(x) = 2^x and g(x) = $(\frac{1}{2})^x$ then the product f(x) . g(x) is

(a) 4^x (b) 0 (c) 1^x (d) 1

12) In a function if the independent variable is acting as an index then the function is known as

(a) exponential function (b) logarithmic function
(c) trigonometric function (d) Inverse function

13) The minimum value of the function f(x) = | x | is

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

14) The slope of the graph of f(x) = $\frac{|x|}{x}$; x>0 is

(a) m=1 (b) m=0 (c) m=-1 (d) m is undefined

15) The greatest integer function f(x) = [x], in the range $3 \leq x < 4$ has the value f(x) = ...

(a) 1 (b) 3 (c) 4 (d) 2

DIFFERENTIAL CALCULUS 7

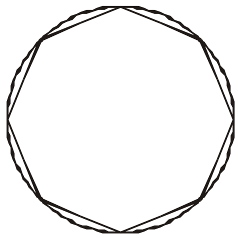
Calculus is the branch of Mathematics that concerns itself with the rate of change of one quantity with respect to another quantity. The foundations of Calculus were laid by Isaac Newton and Gottfried Wilhelm Von Leibnitz.

Calculus is divided into two parts: namely, Differential Calculus and Integral Calculus. In this chapter, we learn what a derivative is, how to calculate it .

7.1 LIMIT OF A FUNCTION

7.1.1 Limiting Process:

The concept of limit is very important for the formal development of calculus. Limiting process can be explained by the following illustration:



Let us inscribe a regular polygon of 'n' sides in a unit circle. Obviously the area of the polygon is less than the area of the unit circle (π sq.units). Now if we increase the number of sides 'n' of the polygon, area of the polygon increases but still it is less than the area of the unit circle. Thus as the number of sides of the polygon increases, the area of the polygon approaches the area of the unit circle.

7.1.2 Limit of a function

Let $f : R \to R$ be a function. We are interested in finding a real number l to which the value f(x) of the function f approaches when x approaches a given number 'a'.

Illustration 1

Let a function f : R→R be defined as f(x) = 2x + 1 as x→3.

x ® 3^+	3.1	3.01	3.001	3.0001	3.00001	. . .
f(x) = 2x + 1	7.2	7.02	7.002	7.0002	7.00002	. . .
| f(x) – 7 |	0.2	0.02	0.002	0.0002	0.00002	. . .

From the above table, we observe that as $x \to 3^+$ (i.e. x→3 from right of 3) f(x)→7. Here 7 is called the right hand limit of f(x) as $x \to 3^+$.

Further,

x®3^-	2.9	2.99	2.999	2.9999	2.99999	…
f(x) = 2x +1	6.8	6.98	6.998	6.9998	6.99998	…
| f(x) – 7 |	0.2	0.02	0.002	0.0002	0.00002	…

From this table, we observe that as $x \to 3^-$ (i.e. x→3 from left of 3) f(x)→7. Here 7 is called the left hand limit of f(x) as $x \to 3^-$.

Thus we find as x→3 from either side, f(x)→7. This means that we can bring f(x) as close to 7 as we please by taking x sufficiently closer to 3 i.e., the difference

|f(x) – 7 | can be made as small as we please by taking x sufficiently nearer to 3.

This is denoted by $\underset{x \to 3}{Lt}\ f(x) = 7$

Illustration 2

Let a function f : R - {2}→R be defined as $\frac{x^2 - 4}{x - 2}$ as x→2.

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
f(x)	3.9	3.99	3.999	3.9999	-	4.0001	4.001	4.01	4.1
|f(x)-4|	0.1	0.01	0.001	0.0001	-	0.0001	0.001	0.01	0.1

From the above table we observe that as x→2 from the left as well as from the right, f(x)→ 4, i.e., difference |f(x)-4| can be made as small as we please by taking x sufficiently nearer to 2. Hence 4 is the limit of f(x) as x approaches 2.

i.e $\underset{x \to 2}{Lt} f(x) = 4$

From the above two illustrations we get that if there exists a real number l such that the difference $|f(x)-l|$ can be made as small as we please by taking x sufficiently close to 'a' (but not equal to a), then l is said to be the limit of f(x) as x approaches 'a'.

It is denoted by $\underset{x \to a}{Lt} f(x) = l$.

Observation :

(i) If we put x = a in f(x), we get the functional value f(a). In general, f(a) ≠ l. Even if f(a) is undefined, the limiting value l of f(x) when x→a may be defined as a finite number.

(ii) The limit f(x) as x tends to 'a' exists if and only if $\underset{x \to a^+}{Lt} f(x)$ and $\underset{x \to a^-}{Lt} f(x)$ exist and are equal.

7.1.3 Fundamental Theorems on Limits

(i) $\underset{x \to a}{Lt} [f(x)+g(x)] = \underset{x \to a}{Lt} f(x) + \underset{x \to a}{Lt} g(x)$

(ii) $\underset{x \to a}{Lt} [f(x) - g(x)] = \underset{x \to a}{Lt} f(x) - \underset{x \to a}{Lt} g(x)$

(iii) $\underset{x \to a}{Lt} [f(x) . g(x)] = \underset{x \to a}{Lt} f(x) . \underset{x \to a}{Lt} g(x)$

(iv) $\underset{x \to a}{Lt} [f(x) / g(x)] = \underset{x \to a}{Lt} f(x) / \underset{x \to a}{Lt} g(x)$, provided $\underset{x \to a}{Lt} g(x) \neq 0$

(v) $\underset{x \to a}{Lt} [c\, f(x)] = c \underset{x \to a}{Lt} f(x)$

7.1.4 Standard results on Limits

(i) $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = n\, a^{n-1}$, n is a rational number.

(ii) $\underset{\theta \to 0}{Lt} \dfrac{\sin\theta}{\theta} = 1$, θ being in radian measure.

(iii) $\underset{x\to 0}{Lt}\dfrac{a^x-1}{x} = \log_e a$

(iv) $\underset{x\to 0}{Lt}\dfrac{e^x-1}{x} = 1$

(v) $\underset{n\to\infty}{Lt}(1 + 1/n)^n = e$

(vi) $\underset{x\to 0}{Lt}(1+x)^{1/x} = e$

(vii) $\underset{x\to 0}{Lt}\dfrac{\log(1+x)}{x} = 1$

Example 1

Evaluate $\underset{x\to 2}{Lt}\dfrac{x^2-4x+6}{x+1}$

Solution:

$$\underset{x\to 2}{Lt}\frac{x^2-4x+6}{x+1} = \frac{\underset{x\to 2}{Lt}(x^2-4x+6)}{\underset{x\to 2}{Lt}(x+1)}$$

$$= \frac{(2)^2-4(2)+6}{2+1} = 2/3$$

Example 2

Evaluate $\underset{x\to p/4}{Lt}\dfrac{3\sin 2x+2\cos 2x}{2\sin 2x-3\cos 2x}$

Solution:

$$\frac{\underset{x\to p/4}{Lt} 3\sin 2x+2\cos 2x}{\underset{x\to p/4}{Lt} 2\sin 2x-3\cos 2x} = \frac{3\sin(p/2)+2\cos(p/2)}{2\sin(p/2)-3\cos(p/2)}$$

$$= \frac{3}{2}$$

Example 3

Evaluate $\underset{x\to 5}{Lt}\dfrac{x^2-25}{x-5}$

Solution:

$$\underset{x\to 5}{Lt}\ \frac{x^2-25}{x-5} = \underset{x\to 5}{Lt}\ \frac{(x+5)(x-5)}{(x-5)}$$

$$= \underset{x\to 5}{Lt}\ (x+5) = 10$$

Example 4

Evaluate $\underset{x\to 0}{Lt}\ \frac{\sqrt{2+3x}-\sqrt{2-5x}}{4x}$

Solution:

$$\underset{x\to 0}{Lt}\ \frac{\sqrt{2+3x}-\sqrt{2-5x}}{4x}$$

$$= \underset{x\to 0}{Lt}\ \left\{\frac{\left(\sqrt{2+3x}-\sqrt{2-5x}\right)\left(\sqrt{2+3x}+\sqrt{2-5x}\right)}{4x\left(\sqrt{2+3x}+\sqrt{2-5x}\right)}\right\}$$

$$= \underset{x\to 0}{Lt}\ \frac{(2+3x)-(2-5x)}{4x\left(\sqrt{2+3x}+\sqrt{2-5x}\right)}$$

$$= \underset{x\to 0}{Lt}\ \frac{8x}{4x\left(\sqrt{2+3x}+\sqrt{2-5x}\right)}$$

$$= \underset{x\to 0}{Lt}\ \frac{2}{\sqrt{2+3x}+\sqrt{2-5x}}$$

$$= \frac{2}{\sqrt{2}+\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Example 5

Evaluate $\underset{x\to a}{Lt}\ \frac{x^{3/5}-a^{3/5}}{x^{1/3}-a^{1/3}}$

Solution:

$$\underset{x\to a}{Lt}\ \frac{x^{3/5}-a^{3/5}}{x^{1/3}-a^{1/3}} = \underset{x\to a}{Lt}\left\{\frac{x^{3/5}-a^{3/5}}{x-a} \div \frac{x^{1/3}-a^{1/3}}{x-a}\right\}$$

$$= \frac{3}{5}\,a^{-2/5} \div \frac{1}{3}\,a^{-2/3} = \frac{9}{5}\,a^{-2/5+2/3} = \frac{9}{5}\,a^{4/15}$$

Example 6

Evaluate $\underset{x\to 0}{Lt}\ \dfrac{\sin 5x}{\sin 3x}$

Solution :

$$\underset{x\to 0}{Lt}\ \frac{\sin 5x}{\sin 3x} = \underset{x\to 0}{Lt}\left\{\frac{5x \times \dfrac{\sin 5x}{5x}}{3x \times \dfrac{\sin 3x}{3x}}\right\}$$

$$= \frac{5}{3}\ \underset{x\to 0}{Lt}\left\{\frac{\dfrac{\sin 5x}{5x}}{\dfrac{\sin 3x}{3x}}\right\} = \frac{5}{3}$$

Example 7

If $\underset{x\to 1}{Lt}\dfrac{x^4-1}{x-1} = \underset{x\to a}{Lt}\ \dfrac{x^3-a^3}{x^2-a^2}$**, find the value of a.**

Solution:

$$\text{LHS} = \underset{x\to 1}{Lt}\frac{x^4-1}{x-1} = 4$$

$$\text{RHS} = \underset{x\to a}{Lt}\ \frac{x^3-a^3}{x^2-a^2}$$

$$= \frac{\underset{x\to a}{Lt}\ \dfrac{x^3-a^3}{x-a}}{\underset{x\to a}{Lt}\ \dfrac{x^2-a^2}{x-a}} = \frac{3a^2}{2a} = \frac{3a}{2}$$

$$\therefore\ 4 = \frac{3a}{2}$$

$$\therefore\ a = \frac{8}{3}$$

Example 8

Evaluate $\underset{x\to\infty}{Lt}\dfrac{6-5x^2}{4x+15x^2}$

Solution:

$$\underset{x\to\infty}{Lt}\frac{6-5x^2}{4x+15x^2} = \underset{x\to\infty}{Lt}\frac{\frac{6}{x^2}-5}{\frac{4}{x}+15}$$

Let y = $\frac{1}{x}$ so that y→0, as x →∞

$$= \underset{y\to 0}{Lt}\frac{6y^2-5}{4y+15}$$

= -5/15 = - 1/3.

Example 9

Show that $\underset{n\to\infty}{Lt}\frac{1^2+2^2+3^2+\ldots+n^2}{n^3} = \frac{1}{3}$

Solution:

$$\underset{n\to\infty}{Lt}\frac{1^2+2^2+3^2+\ldots+n^2}{n^3} = \underset{n\to\infty}{Lt}\frac{n(n+1)(2n+1)}{6n^3}$$

$$= \underset{n\to\infty}{Lt}\frac{1}{6}\left[\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right]$$

$$= \underset{n\to\infty}{Lt}\frac{1}{6}\left[1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

Let y = 1/n so that y→0, as n →∞

$$= \underset{y\to 0}{Lt}\frac{1}{6}\left[(1)(1)(2)\right] = \frac{1}{3}$$

EXERCISE 7.1

1) Evaluate the following limits

(i) $\underset{x\to 2}{Lt}\frac{x^3+2}{x+1}$ (ii) $\underset{x\to p/4}{Lt}\frac{2\sin x+3\cos x}{3\sin x-4\cos x}$

(iii) $\underset{x\to 2}{Lt}\frac{x^2-5x+6}{x^2-7x+10}$ (iv) $\underset{x\to 0}{Lt}\frac{\sqrt{2-x}-\sqrt{2+x}}{x}$

(v) $\underset{x\to 3}{Lt}\left(\frac{x}{x-3}-\frac{9}{x^2-3x}\right)$ (vi) $\underset{q\to 0}{Lt}\ \frac{\tan q}{q}$

(vii) $\underset{x\to a}{Lt}\ \frac{x^{5/8}-a^{5/8}}{x^{1/3}-a^{1/3}}$ (viii) $\underset{x\to 0}{Lt}\ \frac{\sin 5x}{3x}$

(ix) $\underset{x\to\infty}{Lt}\ \frac{x-1}{x+1}$ (x) $\underset{x\to 0}{Lt}\ \frac{\tan 8x}{\sin 2x}$

(xi) $\underset{x\to\infty}{Lt}\ \frac{(3x-1)(4x-2)}{(x+8)(x-1)}$ (xii) $\underset{x\to\infty}{Lt}\ \frac{5x^2+3x-6}{2x^2-5x+1}$

2) If $\underset{x\to 2}{Lt}\ \frac{x^n-2^n}{x-2} = 80$ find n. (where n is a positive integer)

3) Prove that $\underset{x\to 0}{Lt}\ \frac{(1+x)^n-1}{x} = n.$

4) If f(x) = $\frac{x^7-128}{x^5-32}$, find $\underset{x\to 2}{Lt}$ f (x) and f (2), if they exist.

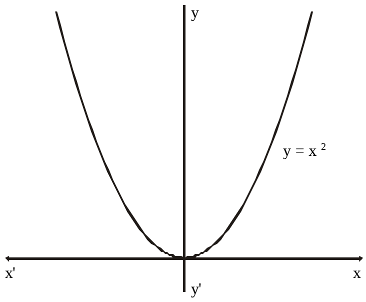
5) If f(x) = $\frac{px+q}{x+1}$, $\underset{x\to 0}{Lt}$ f(x) = 2 and $\underset{x\to\infty}{Lt}$ f(x) = 1, prove that f (-2) = 0

7.2 CONTINUITY OF A FUNCTION

7.2.1 Continuity

In general, a function f(x) is continuous at x = a if its graph has no break at x = a. If there is any break at the point x = a , then we say the function is not continuous at the point x = a. If a function is continuous at all points in an interval it is said to be continuous in the interval.

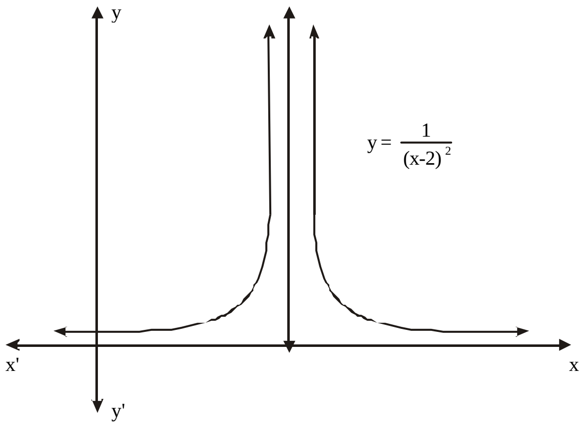
Illustration 1



From the graph we see that the graph of $y = x^2$ has no break. Therefore, it is said to be continuous for all values of x.

Illustration 2

From the graph of $y = \frac{1}{(x-2)^2}$ we see that the graph has a break at x = 2. Therefore it is said to be discontinuous at x = 2.



Definition

A function f(x) is continuous at x = a if

(i) f(a) exists.

(ii) $\underset{x \to a}{Lt}$ f(x) exists

(iii) $\underset{x \to a}{Lt}$ f(x) = f(a).

Observation:

If one or more of the above conditions is not satisfied at a point x = a by the function f(x), then the function is said to be discontinuous at x = a.

7.2.2 Properties of continuous function:

If f(x) and g(x) are two functions which are continuous at x = a then

(i) f(x) + g(x) is continuous at x = a.

(ii) f(x) – g(x) is continuous at x = a.

(iii) f(x) . g(x) is continuous at x = a.

(iv) $\frac{f(x)}{g(x)}$ is continuous at x = a, provided g(a) $\neq$ 0.

(v) If f(x) is continuous at x = a and f(a) $\neq$ 0 then $\frac{1}{f(x)}$ is continuous at x = a.

(vi) If f(x) is continuous at x = a, then |f(x)| is also continuous at x = a.

***Observation*:**

(i) Every polynomial function is continuous.

(ii) Every rational function is continuous.

(iii) Constant function is continuous.

(iv) Identity function is continuous.

Example 10

Let f(x) = $\begin{cases} \frac{\sin 3x}{x} & ; \quad x \neq 0 \\ 1 & ; \quad x = 0 \end{cases}$

Is the function continuous at x = 0 ?

Solution:

Now we shall investigate the three conditions to be satisfied by f(x) for its continuity at x = 0.

(i) f(a) = f(0) = 1 is defined at x = 0.

(ii) $\underset{x \to 0}{Lt}$ f(x) = $\underset{x \to 0}{Lt} \frac{\sin 3x}{x}$ = 3.

(iii) $\underset{x \to 0}{Lt}$ f(x) = 3 $\neq$ f(0) = 1

condition (iii) is not satisfied.

Hence the function is discontinuous at x = 0.

Example 11

Find the points of discontinuity of the function $\frac{x^2 + 6x + 8}{x^2 - 5x + 6}$

Solution:

The points of discontinuity of the function is obtained when the denominator vanishes.

i.e., $x^2 - 5x + 6 = 0$

$\Rightarrow (x - 3)(x - 2) = 0$

$\Rightarrow x = 3; x = 2.$

Hence the points of discontinuity of the function are x = 3 and x = 2.

Example 12

Rs. 10,000 is deposited into a savings account for 3 months at an interest rate 12% compounded monthly. Draw the graph of the account's balance versus time (in months). Where is the graph discontinuous?

Solution :

At the end of the first month the account's balance is

10,000 + 10,000 (.01) = Rs. 10,100.

At the end of the second month, the account's balance is

10,100 + 10,100(.01) = Rs. 10,201.

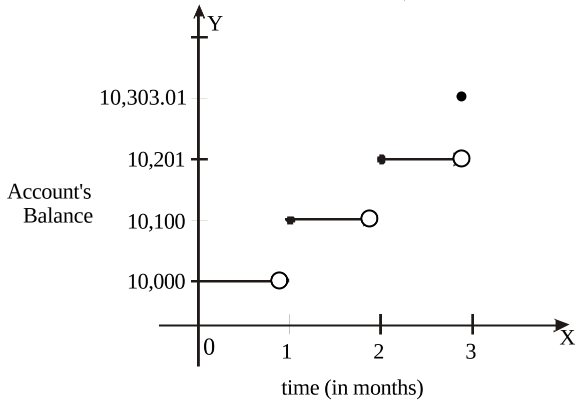
At the end of the third month, the account's balance is

10,201 + 10,201 (.01) = Rs. 10,303.01.

i.e.

X (time)	1	2	3
Y (Balance)	10,100	10,201	10,303.01

The graph of the account's balance versus time, t.



Since the graph has break at t = 1, t = 2, t = 3, it is discontinuous at t = 1, t =2 and t = 3.

Observation:

These discontinuities occur at the end of each month when interest is computed and added to the account's balance.

EXERCISE 7.2

1) Prove that cos x is continuous

2) Find the points of discontinuity of the function $\dfrac{2x^2+6x-5}{12x^2+x-20}$

3) Show that a constant function is always continuous.

4) Show that f(x) = | x | is continuous at the origin.

5) Prove that $f(x) = \dfrac{x+2}{x-1}$ is discontinuous at x = 1.

6) Locate the points of discontinuity of the function $\dfrac{x+2}{(x-3)(x-4)}$

7.3 CONCEPT OF DIFFERENTIATION

7.3.1 Differential coefficient

Let y denote the function f(x). Corresponding to any change in the value of x there will be a corresponding change in the value of y. Let Δx denote the increment in x. The corresponding increment in y is denoted by Δy. Since

$$y = f(x)$$

$$y+\Delta y = f(x+\Delta x)$$

$$\Delta y = f(x+\Delta x) - f(x)$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$\dfrac{\Delta y}{\Delta x}$ is called the incremental ratio.

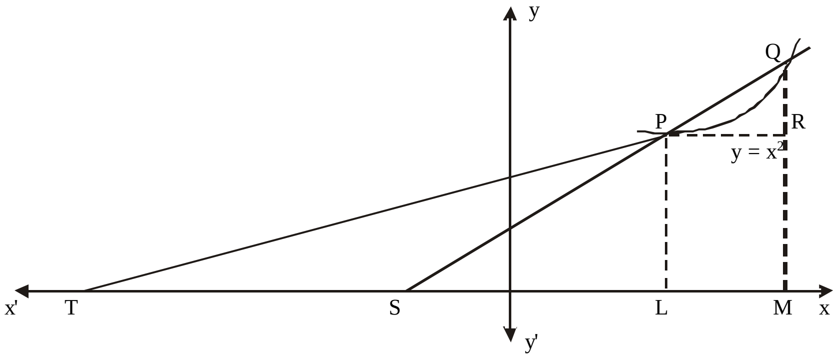
Now $\underset{\Delta x \to 0}{Lt} \dfrac{\Delta y}{\Delta x}$ is called the differential coefficient (or derivative) of y with respect to x and is denoted by $\dfrac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$$

The process of obtaining the differential coefficient (or derivative) is called differentiation. The notations y_1 , $f'(x)$, D(f (x)) are used to denote the differential coefficient of f(x) with respect to x.

7.3.2 Geometrical interpretation of a derivative.

Let P (a , f (a)) and Q (a + h , f (a + h)) be the two points on the curve y = f (x)



Draw the ordinate PL, QM and draw PR $\perp$ MQ.

we have

$$PR = LM = h$$

and

$$QR = MQ - LP = f(a+h) - f(a)$$

$$\frac{QR}{PR} = \frac{f(a+h) - f(a)}{h}$$

As Q→P along the curve, the limiting position of PQ is the tangent PT to the curve at the point P . Also as Q→P along the curve, h→0

$$\text{Slope of the tangent PT} = \underset{Q \to P}{Lt} \text{ (slope of PQ)}$$

$$= \underset{h \to 0}{Lt} \frac{f(a+h) - f(a)}{h}$$

$\therefore$ The derivative of f at a is the slope of the tangent to the curve y = f (x) at the point (a, f(a))

7.3.3 Differentiation from first principles.

The method of finding the differential coefficient of a function y = f(x) directly from the definition is known as differentiation from first principles or ab- initio. This process consists of following five steps.

Step (i) Equating the given function to y i.e., y = f(x)

Step (ii) In the given function replace x by $x + \Delta x$ and calculate the new value of the function $y + \Delta y$.

Step (iii) Obtain $\Delta y = f(x + \Delta x) - f(x)$ and simplify Δy.

Step (iv) Evaluate $\frac{\Delta y}{\Delta x}$

Step (v) Find $\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$

7.3.4 Derivatives of standard functions using first principle

(i) Derivative of x^n, where n is any rational number.

Proof :

Let $y = x^n$

Let Δx be a small arbitrary increment in x and Δy be the corresponding increment in y.

$$\therefore \quad y + \Delta y = (x + \Delta x)^n$$

$$\Delta y = (x + \Delta x)^n - y$$

$$= (x + \Delta x)^n - x^n$$

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$\therefore \quad \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt} \frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x}$$

$$\therefore \quad \frac{dy}{dx} = \underset{(x+\Delta x)\to x}{Lt} \frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x} \quad \text{as} \quad \Delta x \to 0, \quad x + \Delta x \to x$$

$$= n x^{n-1} \qquad \left(\because \underset{x\to a}{Lt} \frac{x^n - a^n}{x - a} = n a^{n-1} \right)$$

$$\frac{d}{dx}\left(x^n\right) = n x^{n-1}$$

(ii) Derivative of sinx

Let y = sinx

Let Δx be a small increment in x and Δy be the corresponding increment in y.

Then y + Δy = sin(x + Δx)

Δy = sin(x + Δx) – y

= sin(x +Δx) – sinx

$$\frac{\Delta y}{\Delta x} = \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$= \frac{2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \cos\left(x + \frac{\Delta x}{2}\right) . \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x\to 0}{Lt} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x\to 0}{Lt} \cos(x + \Delta x/2) . \underset{\Delta x\to 0}{Lt} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$= \cos x \underset{\frac{\Delta x}{2}\to 0}{Lt} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$= (\cos x).1 \qquad \left(\because \underset{q\to 0}{Lt} \frac{\sin q}{q} = 1\right)$$

$$= \cos x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

(iii) Derivative of e^x

Let $y = e^x$

Let Δx be a small arbitrary increment in x and Δy be the corresponding increment in y.

$$\text{Then } y + \Delta y = e^{x+\Delta x}$$

$$\Delta y = e^{x+\Delta x} - y$$

$$\Delta y = e^{x+\Delta x} - e^x$$

$$= e^x (e^{\Delta x} - 1)$$

$$\frac{\Delta y}{\Delta x} = \frac{e^x (e^{\Delta x} - 1)}{\Delta x}$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x\to 0}{Lt} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x\to 0}{Lt} \frac{e^x (e^{\Delta x} - 1)}{\Delta x}$$

$$= e^x \underset{\Delta x\to 0}{Lt} \frac{(e^{\Delta x} - 1)}{\Delta x}$$

$$= e^x\, 1 \left(\text{since } \underset{h\to 0}{Lt} \frac{(e^h - 1)}{h} = 1\right)$$

$$= e^x$$

$$\therefore \frac{d}{dx}(e^x) = e^x$$

(iv) Derivative of log x

Let y = log x

Let Δx be a small increment in x and Δy be the corresponding increment in y.

Then $y + \Delta y = \log (x + \Delta x)$

$\Delta y = \log (x + \Delta x) - y$

$= \log (x + \Delta x) - \log x$

$$\Delta y = \log_e \left(\frac{x + \Delta x}{x} \right)$$

$$= \log_e \left(1 + \frac{\Delta x}{x} \right)$$

$$\frac{\Delta y}{\Delta x} = \frac{\log_e \left(1 + \frac{\Delta x}{x} \right)}{\Delta x}$$

$$\therefore \quad \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt} \frac{\log_e \left(1 + \frac{\Delta x}{x} \right)}{\Delta x}$$

put $\frac{\Delta x}{x} = h$

$\therefore \quad \Delta x = hx$ and as $\Delta x \to 0$, $h \to 0$.

$$\therefore \quad \frac{dy}{dx} = \underset{h \to 0}{Lt} \frac{\log (1 + h)}{hx}$$

$$= \frac{1}{x} \underset{h \to 0}{Lt} \frac{\log_e (1 + h)}{h}$$

$$= \frac{1}{x} \underset{h \to 0}{Lt} \log (1 + h)^{\frac{1}{h}}$$

$$= \frac{1}{x} \, 1$$

$$= \frac{1}{x} \quad (\because \underset{h \to 0}{Lt} \log (1 + h)^{\frac{1}{h}} = 1)$$

$$\therefore \frac{d}{dx}(\log x) = \frac{1}{x}$$

Observation :

$$\frac{d}{dx}(\log\ x) = \frac{1}{x} \underset{h\to 0}{Lt} \log(1+h)^{\frac{1}{h}}$$

$$= \frac{1}{x}\log_e e$$

(v) Derivative of a constant

Let y = k, where k is constant.

Let Δx be a small increment in x and Δy be the corresponding increment in y.

Then $y + \Delta y = k$

$$\Delta y = k - y$$

$$= k - k$$

$$\Delta y = 0$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = 0$$

$$\therefore \quad \frac{dy}{dx} = \underset{\Delta x\to 0}{Lt} \frac{\Delta y}{\Delta x} = 0$$

$$\therefore \quad \frac{d}{dx}\text{(any constant)} = 0$$

7.3.5 General Rules for differentiation

Rule 1 **Addition Rule**

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx},$$ where u and v are functions of x

Proof:

Let y = u + v. Let Δx be a small arbitrary increment in x. Then Δu, Δv, Δy are the corresponding increments in u, v and y respectively.

Then $y + \Delta y = (u + \Delta u) + (v + \Delta v)$

$$\Delta y = (u + \Delta u) + (v + \Delta v) - y$$

$$= u + \Delta u + v + \Delta v - u - v.$$

$$\Delta y = \Delta u + \Delta v$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

$$\therefore \quad \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt} \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right)$$

$$= \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} + \underset{\Delta x \to 0}{Lt} \frac{\Delta v}{\Delta x}$$

$$= \frac{du}{dx} + \frac{dv}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Observation :

Obviously this rule can be extended to the algebraic sum of a finite number of functions of x

Rule 2 **Difference rule**

If u and v are differentiable functions of x and y and y = u-v then

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

Rule 3 **Product rule**

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$, where u and v are functions of x

Proof:

Let y = uv where u and v are separate functions of x.

Let Δx be a small increment in x and let Δu, Δv, Δy are the corresponding increments in u, v, and y respectively.

Then $\quad y + \Delta y = (u + \Delta u)(v + \Delta v)$

$$\Delta y = (u + \Delta u)(v + \Delta v) - y$$

$$= (u + \Delta u)(v + \Delta v) - uv$$

$$= u.\Delta v + v\Delta u + \Delta u\,\Delta v$$

$$\therefore \frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt} u\frac{\Delta v}{\Delta x} + \underset{\Delta x \to 0}{Lt} v\frac{\Delta u}{\Delta x} + \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x}\Delta v$$

$$= u\underset{\Delta x \to 0}{Lt} \frac{\Delta v}{\Delta x} + v\underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} + \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} \underset{\Delta x \to 0}{Lt} \Delta v$$

$$= u\frac{dv}{dx} + v\frac{du}{dx} + \frac{du}{dx}(0) \quad (\because \Delta x \to 0, \Delta v = 0)$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Observation Extension of product rule

If $y = uvw$ then

$$\frac{dy}{dx} = uv\frac{d}{dx}(w) + wu\frac{d}{dx}(v) + wv\frac{d}{dx}(u)$$

Rule 4 **Quotient rule**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$, where u and v are functions of x

Proof:

Let $y = \frac{u}{v}$ where u and v are separate functions of x. Let Δx be a small increment in x and $\Delta u, \Delta v, \Delta y$ are the corresponding increments in u,v and y respectively.

$$\text{Then } y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - y$$

$$= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{v(u+\Delta u) - u(v+\Delta v)}{v(v+\Delta v)}$$

$$= \frac{v\,\Delta u - u\,\Delta v}{v(v+\Delta v)}$$

$$\frac{\Delta y}{\Delta x} = \frac{v.\frac{\Delta u}{\Delta x} - u.\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}$$

$$\therefore \quad \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt} \frac{v.\frac{\Delta u}{\Delta x} - u.\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2 + 0} \quad \text{(Since } \Delta x \to 0, \Delta v = 0\text{)}$$

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

Rule 5 **Derivative of a scalar Product of a function:**

$$\frac{d}{dx}[c\ f(x)] = c\frac{d}{dx}[f(x)] \text{ , where c is constant.}$$

Proof:

Let $y = c\, f(x)$

Let Δx be a small increment in x and Δy be the corresponding increment in y.

Then $y + \Delta y = c\, f(x + \Delta x)$

$$\Delta y = cf(x + \Delta x) - y$$

$$\Delta y = cf(x + \Delta x) - c\, f(x)$$

$$= c(f(x + \Delta x) - f(x))$$

$$\frac{\Delta y}{\Delta x} = \frac{c\big(f(x + \Delta x) - f(x)\big)}{\Delta x}$$

$$\therefore \quad \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt} \frac{c(f(x+\Delta x) - f(x))}{\Delta x}$$

$$= c\, f^{|}(x)$$

$$\therefore \quad \frac{d}{dx} (cf(x)) = c\, f^{|}(x)$$

Standard results

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(iii) $\frac{d}{dx}(x) = 1$

(iv) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

(v) $\frac{d}{dx}(kx) = k$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\operatorname{cosec} x) = -\cot x \,.\operatorname{cosec} x$

(x) $\frac{d}{dx}(\sec x) = \sec x \,.\tan x$

(xi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(xii) $\frac{d}{dx}(e^x) = e^x$

(xiii) $\frac{d}{dx}\left(e^{ax+b}\right) = a\ e^{ax+b}$

(xiv) $\frac{d}{dx}$ (log x) $= \frac{1}{x}$

(xv) $\frac{d}{dx}$[log (x + a)] $= \frac{1}{x+a}$

(xvi) $\frac{d}{dx}$ (Constant) = 0.

Example 13

Differentiate $6x^4 - 7x^3 + 3x^2 - x + 8$ with respect to x.

Solution:

Let $y = 6x^4 - 7x^3 + 3x^2 - x + 8$

$$\frac{dy}{dx} = \frac{d}{dx}(6x^4) - \frac{d}{dx}(7x^3) + \frac{d}{dx}(3x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(8)$$

$$= 6\frac{d}{dx}(x^4) - 7\frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(8)$$

$$= 6(4x^3) - 7(3x^2) + 3(2x) - (1) + 0$$

$$\frac{dy}{dx} = 24x^3 - 21x^2 + 6x - 1$$

Example 14

Find the derivative of $3x^{2/3} - 2\log_e x + e^x$

Solution:

Let $y = 3x^{2/3} - 2\log_e x + e^x$

$$\frac{dy}{dx} = 3\frac{d}{dx}(x^{2/3}) - 2\frac{d}{dx}(\log_e x) + \frac{d}{dx}(e^x)$$

$$= 3(2/3)x^{-1/3} - 2(1/x) + e^x$$

$$= 2x^{-1/3} - 2/x + e^x$$

Example 15

If y = cos x + tan x , find $\frac{dy}{dx}$ at x = $\frac{ð}{6}$

Solution:

y = cos x + tan x

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x) + \frac{d}{dx}(\tan x)$$

$$= -\sin x + \sec^2 x$$

$$\frac{dy}{dx} \text{ (at } x = \frac{p}{6}\text{)} = -\sin\frac{p}{6} + (\sec \pi/6)^2$$

$$= -\frac{1}{2} + \frac{4}{3} = \frac{5}{6}$$

Example 16

Differentiate : cosx . logx with respect to x

Solution:

Let $y = \cos x \,.\, \log x$

$$\frac{dy}{dx} = \cos x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\cos x)$$

$$= \cos x \frac{1}{x} + (\log x)(-\sin x)$$

$$= \frac{\cos x}{x} - \sin x \log x$$

Example 17

Differentiate $x^2 e^x \log x$ with respect to x

Solution:

Let $y = x^2 e^x \log x$

$$\frac{dy}{dx} = x^2 e^x \frac{d}{dx}(\log x) + x^2 \log x \frac{d}{dx}(e^x) + e^x \log x \frac{d}{dx}(x^2)$$

$$= (x^2 e^x)(1/x) + x^2 \log x\,(e^x) + e^x \log x\,(2x)$$

$$= x e^x + x^2 e^x \log x + 2x e^x \log x$$

$$= x e^x (1 + x \log x + 2 \log x)$$

Example 18

Differentiate $\frac{x^2 + x + 1}{x^2 - x + 1}$ with respect to x

Solution:

Let $y = \frac{x^2 + x + 1}{x^2 - x + 1}$

$$\frac{dy}{dx} = \frac{(x^2 - x + 1)\frac{d}{dx}(x^2 + x + 1) - (x^2 + x + 1)\frac{d}{dx}(x^2 - x + 1)}{(x^2 - x + 1)^2}$$

$$= \frac{(x^2 - x + 1)(2x + 1) - (x^2 + x + 1)(2x - 1)}{(x^2 - x + 1)^2}$$

$$= \frac{2(1 - x^2)}{(x^2 - x + 1)^2}$$

EXERCISE 7.3

1) Find from the first principles the derivative of the following functions.
(i) cosx (ii) tanx (iii) cosecx (iv) $\sqrt{x}$

2) Differentiate the following with respect to x.

(i) $3x^4 - 2x^3 + x + 8$

(ii) $\frac{5}{x^4} - \frac{2}{x^3} + \frac{5}{x}$

(iii) $\sqrt{x} + \frac{1}{\sqrt[3]{x}} + e^x$

(iv) $\frac{3 + 2x - x^2}{x}$

(v) tanx + logx

(vi) $x^3 e^x$

(vii) $\frac{3x^3 - 4x^2 + 2}{\sqrt{x}}$

(viii) $ax^n + \frac{b}{x^n}$

(ix) $(x^2 + 1)(3x^2 - 2)$

(x) $(x^2 + 2)\sin x$

(xi) secx tanx

(xii) $x^2 \sin x + 2x \sin x + e^x$

(xiii) $(x^z - x + 1)(x^2 + x + 1)$

(xiv) $x^n \log x$

(xv) $x^2 \tan x + 2x \cot x + 2$

(xvi) $\sqrt{x} \cdot \sec x$

(xvii) $\frac{e^x}{1 + e^x}$

(xviii) $\frac{1 - \cos x}{1 + \cos x}$

(xix) $\frac{3 - 5x}{3 + 5x}$

(xx) $\log\left(e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right)$

(xxi) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

(xxii) $x^2 \log x$

(xxiii) x tanx + cos x

(xxiv) $\frac{e^x}{(1 + x)}$

7.3.6 Derivative of function of a function - Chain Rule.

If y is a function of u and u is a function of x , then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

If y is a function of u , u is a function of v and v is a function of x , then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$ and so on

Example 19

Differentiate with respect to x

(i) $\sqrt{(\sin x)}$ **(ii)** $e^{\sqrt{x}}$

Solution:

(i) $y = \sqrt{(\sin x)}$ put sinx = u

$y = \sqrt{u}$

$\frac{dy}{du} = \frac{1}{2} u^{-1/2}$ and $\frac{du}{dx} = \cos x$

$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \frac{1}{2} u^{-1/2} \cos x$

$= \frac{\cos x}{2\sqrt{(\sin x)}}$

(ii) $y = e^{\sqrt{x}}$

$\frac{dy}{dx} = \frac{d}{dx}(e^{\sqrt{x}})$

$= e^{\sqrt{x}}\frac{d}{dx}(\sqrt{x})$

$= \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

Example 20

Differentiate **log** $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ **with respect to x**

Solution:

Let $y = \log \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$y = \log(e^x + e^{-x}) - \log(e^x - e^{-x})$

$$\frac{dy}{dx} = \frac{d}{dx}\{\log(e^x + e^{-x})\} - \frac{d}{dx}\{\log(e^x - e^{-x})\}$$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}} - \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$= \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x + e^{-x})(e^x - e^{-x})}$$

$$= \frac{e^{2x} - 2 + e^{-2x} - e^{2x} - 2 - e^{-2x}}{e^{2x} - e^{-2x}}$$

$$= \frac{-4}{e^{2x} - e^{-2x}}$$

Example 21

Differentiate log (logx) with respect to x

Solution:

Let $y = \log(\log x)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\log(\log x)\}$$

$$= \frac{1}{\log x} \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x \log x}$$

Example 22

Differentiate e^{4x} sin 4x with respect to x

Solution:

Let $y = e^{4x} \sin 4x$

$$\frac{dy}{dx} = e^{4x}\frac{d}{dx}(\sin 4x) + \sin 4x \frac{d}{dx}(e^{4x})$$
$$= e^{4x}(4\cos 4x) + \sin 4x\,(4e^{4x})$$
$$= 4e^{4x}(\cos 4x + \sin 4x)$$

EXERCISE 7.4

Differentiate the following functions with respect to x

1) $\sqrt{3x^2 - 2x + 2}$

2) $(8 - 5x)^{2/3}$

3) $\sin(e^x)$

4) $e^{\sec x}$

5) $\log \sec x$

6) e^{x^2}

7) $\log(x + \sqrt{(x^2 + 1)})$

8) $\cos(3x - 2)$

9) $\log \cos x^2$

10) $\log\{e^{2x}\sqrt{(x-2)/(x+2)}\}$

11) $e^{\sin x + \cos x}$

12) $e^{\cot x}$

13) $\log\{(e^x/(1 + e^x)\}$

14) $\log(\sin^2 x)$

15) $e^{\sqrt{\tan x}}$

16) $\sin x^2$

17) $\{\log(\log(\log x)\}^n$

18) $\cos^2 x$

19) $e^{-x}\log(e^x + 1)$

20) $\log\{(1 + x^2)/(1 - x^2)\}$

21) $\sqrt[3]{x^3 + x + 1}$

22) $\sin(\log x)$

23) $x^{\log(\log x)}$

24) $(3x^2 + 4)^3$

7.3.7 Derivative of Inverse Functions

If y = f(x) is a differentiable function of x such that the inverse function $x = f^{-1}(y)$ is defined then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \text{ provided } \frac{dy}{dx} \neq 0$$

Standard Results

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{(1+x^2)}$

(iv) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(v) $\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{(1+x^2)}$

Example 23

Differentiate : $\cos^{-1}(4x^3 - 3x)$ with respect to x

Solution :

Let $y = \cos^{-1}(4x^3 - 3x)$

Put $x = \cos\theta$

then $y = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$= \cos^{-1}(\cos 3\theta)$

$y = 3\theta$

$\therefore\ y = 3\cos^{-1} x$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$$

Example 24

Differentiate $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ with respect to x

Let $y = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

Put $x = \tan\theta$

$\therefore\ y = \tan^{-1}\left(\frac{1-\tan q}{1+\tan q}\right)$

$= \tan^{-1}\left(\frac{\tan p/4 - \tan J}{1+\tan p/4 \tan q}\right)$

$= \tan^{-1}\left(\tan\left(\frac{p}{4} - q\right)\right)$

$$y = \frac{p}{4} - q$$

$$y = \frac{p}{4} - \tan^{-1} x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{1+x^2}$$

7.3.8 Logarithmic Differentiation

Let y = f (x) be a function. The process of taking logarithms on both sides and differentiating the function is called logarithmic differentiation.

Example 25

Differentiate $\frac{(2x+1)^3}{(x+2)^2(3x-5)^5}$ **with respect to x**

Solution:

Let $y = \frac{(2x+1)^3}{(x+2)^2(3x-5)^5}$

$$\log y = \log\left\{\frac{(2x+1)^3}{(x+2)^2(3x-5)^5}\right\}$$

$$= 3\log(2x+1) - 2\log(x+2) - 5\log(3x-5)$$

Differentiating with respect to x,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{3}{2x+1}(2) - 2\frac{1}{x+2}(1) - 5\frac{1}{3x-5} \cdot 3$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{6}{2x+1} - \frac{2}{x+2} - \frac{15}{3x-5}$$

$$\frac{dy}{dx} = y\left[\frac{6}{2x+1} - \frac{2}{x+2} - \frac{15}{3x-5}\right]$$

$$= \frac{(2x+1)^3}{(x+2)^2(3x-5)^5}\left[\frac{6}{2x+1} - \frac{2}{x+2} - \frac{15}{3x-5}\right]$$

Example 26

Differentiate $(\sin x)^{\cos x}$ **with respect to x**

Solution :

Let $y = (\sin x)^{\cos x}$

Taking logarithms on both sides

$\log y = \cos x \log \sin x$

Differentiating with respect to x

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \frac{d}{dx}(\log \sin x) + \log \sin x . \frac{d}{dx}(\cos x)$$

$$= \cos x \frac{1}{\sin x} . \cos x + \log \sin x (-\sin x)$$

$$= \frac{\cos^2 x}{\sin x} - \sin x \log \sin x$$

$$\frac{dy}{dx} = y[\cot x \cos x - \sin x \log \sin x]$$

$$= (\sin x)^{\cos x} [\cot x \cos x - \sin x \log \sin x]$$

EXERCISE 7.5

Differentiate the following with respect to x

1) $\sin^{-1}(3x - 4x^3)$

2) $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

3) $\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$

4) $\text{Sin}^{-1} \frac{2x}{1 + x^2}$

5) $\tan^{-1} \frac{2x}{1 - x^2}$

6) $\tan^{-1}\left(\frac{\sqrt{1 + x^2} - 1}{x}\right)$

7) $\cot^{-1} \sqrt{1 + x^2} - x$

8) $\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$

9) x^x

10) $(\sin x)^{\log x}$

11) $x^{\sin^{-1} x}$

12) $(3x - 4)^{x-2}$

13) e^{x^x}

14) $x^{\log x}$

15) $\sqrt[3]{\frac{4 + 5x}{4 - 5x}}$

16) $(x^2 + 2)^5 \ (3x^4 - 5)^4$

17) $x^{\frac{1}{x}}$

18) $(\tan x)^{\cos x}$

19) $\left(1+\frac{1}{x}\right)^{x}$

20) $\sqrt{\frac{1+x^2}{1-x^2}}$

21) $\frac{x^3\sqrt{x^2+5}}{(2x+3)^2}$

22) a^x

23) $x^{\sqrt{x}}$

24) $(\sin x)^x$

7.3.9 Derivative of Implicit Functions

The functions of the type y = f (x) are called explicit functions. The functions of the form f (x,y) = c where c is constant are called implicit functions.

Example 27

If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$

Solution:

$x^m y^n = (x+y)^{m+n}$

Taking logarithms,

m log x + n log y = (m + n) log (x + y)

Differentiating with respect to x,

$$\frac{m}{x}+\frac{n}{y}\frac{dy}{dx} = \left(\frac{m+n}{x+y}\right)\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{m}{x}+\frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}+\frac{m+n}{x+y}.\frac{dy}{dx}$$

$$\Rightarrow \frac{n}{y}.\frac{dy}{dx}-\frac{m+n}{x+y}\frac{dy}{dx} = \frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{n}{y}-\frac{m+n}{x+y}\right] = \frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{nx+ny-my-ny}{y(x+y)}\right] = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{nx-my}{y}\right] = \frac{nx-my}{x}$$

$$\therefore \frac{dy}{dx} = \left(\frac{nx - my}{x}\right) \cdot \left(\frac{y}{nx - my}\right)$$

$$= \frac{y}{x}$$

EXERCISE 7.6

Find $\frac{dy}{dx}$ of the following

1) $y^2 = 4ax$

2) $x^2 + y^2 = 9$

3) $xy = c^2$

4) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

5) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

6) $ax^2 + 2hxy + by^2 = 0$

7) $x^2 - 2xy + y^2 = 16$

8) $x^4 + x^2y^2 + y^4 = 0$

9) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

10) $x^y = y^x$

11) $x^2 + y^2 + x + y + l = 0$

12) $y = \cos(x + y)$

13) $x^y = e^{x-y}$

14) $(\cos x)^y = (\sin y)^x$

15) $x^2 - xy + y^2 = 1$

7.3.10 Differentiation of parametric functions

Sometimes variables x and y are given as function of another variable called parameter. We find $\frac{dy}{dx}$ for the parametric functions as given below

Let $x = f(t)$; $y = g(t)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Example 28

If $x = a(q - \sin q)$; $y = a(1 - \cos q)$ find $\frac{dy}{dx}$

Solution:

$$\frac{dx}{d\theta} = a(1-\cos\theta) \quad ; \quad \frac{dy}{d\theta} = a(\sin\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$= \frac{a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$$

EXERCISE 7.7

Find $\frac{dy}{dx}$ for the following functions.

1) $x = a\cos\theta, y = b\sin\theta$

2) $x = ct, \; y = \frac{c}{t}$

3) $x = a\sec\theta, y = b\tan\theta$

4) $3x = t^3, 2y = t^2$

5) $x = a\cos^3\theta, y = a\sin^3\theta$

6) $x = \log t, y = \sin t$

7) $x = e^{\theta}(\sin\theta + \cos\theta); \; y = e^{\theta}(\sin\theta - \cos\theta)$

8) $x = \sqrt{t}, \; y = t + \frac{1}{t}$

9) $x = \cos(\log t) \; ; \; y = \log(\cos t)$

10) $x = 2\cos^2\theta \; ; \; y = 2\sin^2\theta$

11) $x = at^2, \; y = 2at$

7.3.11 Successive Differentiation

Let y be a function of x, and its derivative $\frac{dy}{dx}$ is in general another function of x. Therefore $\frac{dy}{dx}$ can also be differentiated. The derivative of $\frac{dy}{dx}$ namely $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the derivative of the second order. It is written as $\frac{d^2y}{dx^2}$ (or) y_2. Similarly the derivative of $\frac{d^2y}{dx^2}$ namely $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is called the third order derivative and it is written as $\frac{d^3y}{dx^3}$ and so on.

Derivatives of second and higher orders are called higher derivatives and the process of finding them is called Successive differentiation.

Example 29

If $y = e^x \log x$ find y_2

Solution:

$$y = e^x \log x$$

$$y_1 = e^x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(e^x)$$

$$= \frac{e^x}{x} + \log x\,(e^x)$$

$$y_1 = e^x \left(\frac{1}{x} + \log x\right)$$

$$y_2 = e^x \frac{d}{dx}\left(\frac{1}{x} + \log x\right) + \left(\frac{1}{x} + \log x\right)\frac{d}{dx}(e^x)$$

$$y_2 = e^x \left\{-\frac{1}{x^2} + \frac{1}{x}\right\} + \left(\frac{1}{x} + \log x\right)e^x$$

$$= e^x \left\{-\frac{1}{x^2} + \frac{1}{x} + \frac{1}{x} + \log x\right\}$$

$$= e^x \left\{\left(\frac{2x-1}{x^2}\right) + \log x\right\}$$

Example 30

If $x = a(t + \sin t)$ and $y = a(1 - \cos t)$, find $\frac{d^2y}{dx^2}$ at $t = \frac{p}{2}$

Solution:

$$x = a(t + \sin t)\ ; \quad y = a(1 - \cos t)$$

$$\frac{dx}{dt} = a(1 + \cos t)\ ; \quad \frac{dy}{dt} = a \sin t$$

$$= 2a \cos^2 t/2\ ; \quad = 2a \sin t/2 \cos t/2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{2a \sin t/2 \cos t/2}{2a \cos^2 t/2}$$

$$= \tan t/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{2}\sec^2 t/2 \quad \frac{dt}{dx}$$

$$= \frac{1}{2}\sec^2 t/2 \cdot \frac{1}{2a\cos^2 t/2}$$

$$= \frac{1}{4a}\sec^4 t/2$$

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } t = \pi/2} = \frac{1}{4a}(\sec \pi/4)^4$$

$$= \frac{1}{4a}\,4 = \frac{1}{a}$$

Example 31

If $y = \left(x+\sqrt{1+x^2}\right)^m$, **prove that** $(1 + x^2)\, y_2 + x\, y_1 - m^2 y = 0.$

Solution:

$$y = \left(x+\sqrt{1+x^2}\right)^m,$$

$$y_1 = m\left(x+\sqrt{1+x^2}\right)^{m-1}\left\{1+\frac{2x}{2\sqrt{1+x^2}}\right\}$$

$$= m\left(x+\sqrt{1+x^2}\right)^{m-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$= \frac{m\left(x+\sqrt{1+x^2}\right)^m}{\sqrt{1+x^2}}$$

$$y_1 = \frac{my}{\sqrt{1+x^2}}$$

$\Rightarrow (1 + x^2)(y_1)^2 = m^2y^2$

Differentiating with respect to x, we get

$$(1 + x^2).\, 2(y_1)(y_2) + (y_1)^2(2x) = 2m^2 y\, y_1$$

Dividing both sides by $2y_1$, we get

$$(1+x^2)y_2 + xy_1 = m^2y$$

$$\Rightarrow \quad (1+x^2)y_2 + xy_1 - m^2y = 0$$

Example 32

Given $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$; find the value of $\left(\frac{d^2y}{dx^2}\right)$ at the point $t = 2$

Solution:

$$x = t + \frac{1}{t} \quad ; \quad y = t - \frac{1}{t}$$

$$\frac{dx}{dt} = 1 - \frac{1}{t^2} \quad ; \quad \frac{dy}{dt} = 1 + \frac{1}{t^2}$$

$$= \frac{t^2-1}{t^2} \qquad\qquad = \frac{t^2+1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{t^2+1}{t^2} \cdot \frac{t^2}{t^2-1} = \frac{t^2+1}{t^2-1}$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{t^2+1}{t^2-1}\right)$$

$$= \left\{\frac{(t^2-1)2t - (t^2+1)(2t)}{(t^2-1)^2}\right\} \frac{dt}{dx}$$

$$= \left\{\frac{-4t}{(t^2-1)^2}\right\} \frac{t^2}{(t^2-1)}$$

$$= \frac{-4t^3}{(t^2-1)^3}$$

$$\left(\frac{d^2y}{dx^2}\right) \text{ at } t = 2 \quad = \frac{-4(2)^3}{(4-1)^3}$$

$$= \frac{-32}{27}$$

EXERCISE 7.8

1) Find $\frac{d^2y}{dx^2}$, when $y = (4x-1)^2$

2) If $y = e^{-ax}$, find $\frac{d^2y}{dx^2}$

3) If $y = \log(x + 1)$, find $\frac{d^2y}{dx^2}$

4) If $x = at^2$, $y = 2at$ find y_2.

5) Find y_2, when $x = a\cos\theta$, $y = b\sin\theta$.

6) For the parametric equations $x = a\cos^3\theta$, $y = a\sin^3\theta$, find $\frac{d^2y}{dx^2}$

7) If $y = Ae^{ax} - Be^{-ax}$ prove that $\frac{d^2y}{dx^2} = a^2y$

8) If $y = x^2 \log x$ Show that $\frac{d^2y}{dx^2} = 3 + 2\log x$.

9) Prove that $(1 - x^2)y_2 - xy_1 - y = 0$, if $y = e^{\sin^{-1}x}$

10) Show that $x^2y_2 + xy_1 + y = 0$, if $y = a\cos(\log x) + b\sin(\log x)$

11) When $y = \log x$ find $\frac{d^2y}{dx^2}$

EXERCISE 7.9

Choose the correct answer

1) $\underset{x\to 2}{Lt} \frac{2x^2 + x + 1}{x + 2}$ is equal to

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{11}{4}$ (d) 0

2) $\underset{x \to 2}{Lt} \dfrac{2x^2 - x - 1}{x^2 + x - 1}$ is equal to

(a) 0 (b) 1 (c) 5 (d) 2

3) $\underset{x \to 1}{Lt} \dfrac{x^m - 1}{x^n - 1}$ is

(a) mn (b) m + n (c) m - n (d) $\dfrac{m}{n}$

4) $\underset{x \to \infty}{Lt} \dfrac{(x-2)(x+4)}{x(x-9)}$ is equal to

(a) 1 (b) 0 (c) 9 (d) -4

5) $\underset{x \to \infty}{Lt}$ [(1/x) + 2] is equal to

(a) ∞ (b) 0 (c) 1 (d) 2

6) $\underset{x \to \infty}{Lt} \dfrac{1+2+3+...+n}{2n^2+6}$ is

(a) 2 (b) 6 (c) $\dfrac{1}{4}$ (d) $\dfrac{1}{2}$

7) $\underset{x \to p/2}{Lt} \dfrac{\sin x}{x}$ =

(a) π (b) $\dfrac{\pi}{2}$ (c) $\dfrac{2}{\pi}$ (d) None of these

8) If $f(x) = \dfrac{x^2 - 36}{x - 6}$, then f(x) is defined for all real values of x except when x is equal to

(a) 36 (b) 6 (c) 0 (d) None of these

9) The point of discontinuity for the function $\dfrac{2x^2 - 8}{x - 2}$ is

(a) 0 (b) 8 (c) 2 (d) 4

10) A function f(x) is said to be continuous at x = a if $\underset{x \to a}{Lt}$ f(x) =

(a) f (a) (b) f (-a) (c) 2 f (a) (d) f(1/a)

11) The derivative of $2\sqrt{x}$ with respect to x is

(a) $\sqrt{x}$ (b) $1/2\sqrt{x}$ (c) $1/\sqrt{x}$ (d) $1/4\sqrt{x}$

12) $\frac{d}{dx}\left(\frac{1}{x}\right)$ is

(a) log x (b) $1/x^2$ (c) $-(1/x^2)$ (d) $-(1/x)$

13) If $y = 2^x$, then $\frac{dy}{dx}$ is equal to

(a) $2^x \log 2$ (b) 2^x (c) $\log 2^x$ (d) x log 2

14) If $f(x) = x^2 + x + 1$, then $f'(0)$ is

(a) 0 (b) 3 (c) 2 (d) 1

15) $\frac{d}{dx}\left(\frac{1}{x^3}\right)$ is

(a) $-\frac{3}{x^4}$ (b) $-(1/x^3)$ (c) $-(1/x^4)$ (d) $-(2/x^2)$

16) f(x) = cos x + 5 , then $f'(\pi/2)$ is

(a) 5 (b) -1 (c) 1 (d) 0

17) If $y = 5e^x - 3 \log x$ then $\frac{dy}{dx}$ is

(a) $5e^x - 3x$ (b) $5e^x - 3/x$ (c) $e^x - 3/x$ (d) $5e^x - 1/x$

18) $\frac{d}{dx}\left(e^{\log x}\right)$ is

(a) log x (b) $e^{\log x}$ (c) 1/x (d) 1

19) If $y = \sqrt{\sin x}$, then $\frac{dy}{dx} =$

(a) $\frac{\cos x}{2\sqrt{\sin x}}$ (b) $\frac{\sin x}{2\sqrt{\cos x}}$ (c) $\frac{\cos x}{\sqrt{\sin x}}$ (d) $\frac{\cos x}{\sin\sqrt{x}}$

20) $\frac{d}{dx}(e^{4x}) =$

(a) e^{4x} (b) $4e^{4x}$ (c) e^x (d) $4e^{4x-1}$

21) $\frac{d}{dx}(\sin^2 x) =$

(a) 2 sin x (b) sin 2x (c) 2 cos x (d) cos 2x

22. $\frac{d}{dx}(\log \sec x) =$

(a) sec x (b) 1/secx (c) tan x (d) sec x tan x

23) If $y = 2^{-x}$, then $\frac{dy}{dx}$ is equal to

(a) 2^{x-1} (b) $2^{x}\log 2$ (c) $2^{-x}\log(1/2)$ (d) $2^{-x}\log 4$

24) $\frac{d}{dx}(\tan^{-1} 2x)$ is

(a) $\frac{1}{1+x^2}$ (b) $\frac{2}{1+4x^2}$ (c) $\frac{2x^2}{1+4x^2}$ (d) $\frac{1}{1+4x^2}$

25) If $y = e^{ax^2}$, then $\frac{dy}{dx}$ is

(a) 2axy (b) 2ax (c) $2ax^2$ (d) 2ay

26) $\frac{d}{dx}(1 + x^2)^2$ is

(a) $2x(1 + x^2)$ (b) $4x(1 + x^2)$ (c) $x(1+x^2)^3$ (d) $4x^2$

27) If $f(x) = \frac{\log x}{x}$ then $f'(e)$ is

(a) 1/e (b) -1 (c) 0 (d) $\frac{1}{e^2}$

28) $\frac{d}{dx}(x \log x)$ is

(a) log x (b) 1 (c) 1+ logx (d) $\frac{\log x}{x}$

29) If $x = \log \sin\theta$; $y = \log \cos\theta$ then $\frac{dy}{dx}$ is

(a) $-\tan^2\theta$ (b) $\tan^2\theta$ (c) $\tan\theta$ (d) $-\cot^2\theta$

30) If y = x and z = 1/ x then $\frac{dy}{dz}$ is

(a) x^2 (b) $-x^2$ (c) 1 (d) $-1/x^2$

31) If $x = t^2$, and $y = 2t$ then $\frac{dy}{dx}$ is

(a) 2t (b) 1/t (c) 1 + 2t (d) 1/2t

32) If $y = e^{2x}$ then $\frac{d^2y}{dx^2}$ is

(a) 2y (b) 4y (c) y (d) 0

33) If $y = \sin mx$ then $\frac{d^2y}{dx^2}$ is

(a) $-m^2y$ (b) m^2y (c) my (d) $-my$

34) If $y = 3x^3 + x^2 + 1$ then $\frac{d^2y}{dx^2}$ is

(a) $18x$ (b) $18x + 1$ (c) $18x + 2$ (d) $3x^2 + 1$

35) If $y = \log \sec x$ then $\frac{d^2y}{dx^2}$ is

(a) $\sec^2 x$ (b) $\tan x$ (c) $\sec x \tan x$ (d) $\cos x$

36) If $y = e^{3x}$ then $\frac{d^2y}{dx^2}$ at $x = 0$ is

(a) 3 (b) 9 (c) 0 (d) 1

37) If $y = x \log x$ then y_2 is

(a) 1 (b) $\log x$ (c) $1/x$ (d) x

38) If $y = \log(\sin x)$ then $\frac{d^2y}{dx^2}$ is

(a) $\tan x$ (b) $\cot x$ (c) $\sec^2 x$ (d) $-\operatorname{cosec}^2 x$

39) If $y = x^4$ then y_3 is

(a) $4x^3$ (b) $12x^2$ (c) 0 (d) $24x$

40) If $y = \log x$ then y_2 is

(a) $1/x$ (b) $-1/x^2$ (c) e^x (d) 1

41) If $y^2 = x$ then $\frac{dy}{dx}$ is

(a) 1 (b) $1/2x$ (c) $1/2y$ (d) $2y$

42) $\frac{d}{dx}\left(x^a\right)$ is ($a \neq 0$)

(a) $a x^{a-1}$ (b) ax (c) 0 (d) x^{a-1}

43) $\frac{d}{dx}(a^a)$ where $a \neq 0$ is

(a) 0 (b) $a a^{a-1}$ (c) 1 (d) $a \log a$

44) $\frac{d}{dx}(\log \sqrt{x})$ is

(a) $1/\sqrt{x}$ (b) $1/2x$ (c) $1/x$ (d) $1/2\sqrt{x}$

INTEGRAL CALCULUS 8

In this second part of the calculus section we shall study about another process of calculus called Integration. Integration has several applications in Science and Technology as well as in other fields like Economics and Commerce.

8.1 CONCEPT OF INTEGRATION

In chapter 7 we have dealt with the process of derivatives of functions f(x). Generally $f'(x)$ will be another function of x. In this chapter, we will perform an operation that is the reverse process of differentiation. It is called 'anti differentiation' or Integration.

$$\text{If} \quad \frac{d}{dx}[F(x)] = f(x)$$

F(x) is called 'the integral of f(x) and that is represented symbolically as

$$F(x) = \int f(x)\, dx$$

The symbol "$\int$" is the sign of integration and the above statement is read as 'integral of f(x) with respect to x' or 'integral f(x) dx'. f(x) is called 'integrand'.

Generally $\int f(x)\, dx = F(x) + C$, where C is an arbitrary constant.

$\int f(x)\, dx$ is called indefinite integral.

8.2 INTEGRATION TECHNIQUES

Standard results

(i) $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$, provided $n \neq -1$

(ii)	$\int \frac{1}{x^n}\,dx$	=	$\frac{x^{-n+1}}{-n+1}$ + C , provided $n \neq 1$
(iii)	$\int \frac{1}{x}\,dx$	=	log x + C
(iv)	$\int \frac{dx}{x+a}$	=	log (x+a) + C
(v)	$\int k.f(x)\,dx$	=	$k\int f(x)dx + C$
(vi)	$\int k.\,dx$	=	kx + C
(vii)	$\int e^x\,dx$	=	$e^x + C$
(viii)	$\int a^x\,dx$	=	$\frac{a^x}{\log_e a} + C$
(ix)	$\int \sin x\,dx$	=	- cosx + C
(x)	$\int \cos x\,dx$	=	sinx + C
(xi)	$\int \sec^2 x\,dx$	=	tanx + C
(xii)	$\int \sec x \tan x\,dx$	=	secx + C
(xiii)	$\int \operatorname{cosec}^2 x\,dx$	=	- cotx + C
(xiv)	$\int \cot x \operatorname{cosec} x\,dx$	=	- cosecx + C
(xv)	$\int [f_1(x) \pm f_2(x)]\,dx$	=	$\int f_1(x)\,dx \pm \int f_2(x)\,dx$
(xvi)	$\int \frac{dx}{\sqrt{1-x^2}}$	=	$\sin^{-1} x + C$
(xvii)	$\int \frac{dx}{1+x^2}$	=	$\tan^{-1} x + C$
(xviii)	$\int \frac{dx}{x\sqrt{x^2-1}}$	=	$\sec^{-1} x + C$
(xix)	$\int \frac{f'(x)}{f(x)}\,dx$	=	log f(x) + C
(xx)	$\int [f(x)]^n f'(x)dx$	=	$\frac{[f(x)]^{n+1}}{n+1} + C$

Example 1

Evaluate $\int \left(x - \frac{1}{x} \right)^2 dx$

Solution:

$$\int \left(x - \frac{1}{x} \right)^2 dx \quad = \int \left(x^2 - 2 + \frac{1}{x^2} \right) dx$$

$$= \int (x^2 - 2 + x^2)\, dx$$

$$= \frac{x^3}{3} - 2x - \frac{1}{x} + C$$

Example 2

Evaluate $\int \frac{e^x - 2x^2 + xe^x}{x^2 e^x}\, dx$

Solution:

$$\int \frac{e^x - 2x^2 + xe^x}{x^2 e^x}\, dx \quad = \int \left(\frac{e^x}{x^2 e^x} - \frac{2x^2}{x^2 e^x} + \frac{xe^x}{x^2 e^x} \right) dx$$

$$= \int \frac{1}{x^2}\, dx - \int \frac{2}{e^x}\, dx + \int \frac{1}{x}\, dx$$

$$= \int x^{-2}\, dx - 2\int e^{-x}\, dx + \int \frac{1}{x}\, dx$$

$$= \frac{x^{-2+1}}{-2+1} + 2e^{-x} + \log x + c$$

$$= -\frac{1}{x} + 2e^{-x} + \log x + c$$

Example 3

Evaluate $\int \frac{x+1}{\sqrt{x+2}}\, dx$

Solution :

$$\int \frac{x+1}{\sqrt{x+2}}\, dx \quad = \quad \int \frac{x+2}{\sqrt{x+2}}\, dx - \int \frac{dx}{\sqrt{x+2}}$$

(adding and substracting 1 in the numerator)

$$= \int \sqrt{x+2}\, dx - \int \frac{dx}{\sqrt{x+2}}$$

$$= \int (x+2)^{\frac{1}{2}}\, dx - \int (x+2)^{-\frac{1}{2}}\, dx$$

$$= \frac{2}{3}(x+2)^{\frac{3}{2}} - 2(x+2)^{\frac{1}{2}} + C$$

$$= 2(x+2)^{\frac{1}{2}}\left[\frac{(x+2)}{3} - 1\right] + C$$

$$= \frac{2}{3}(x+2)^{\frac{1}{2}}(x-1) + C$$

Example 4

Evaluate $\int \sqrt{1+\sin 2x}\; dx$

Solution:

$$\int \sqrt{1+\sin 2x}\; dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\; dx$$

$$= \int \sqrt{(\sin x + \cos x)^2}\; dx$$

$$= \int (\sin x + \cos x)\; dx$$

$$= (\sin x - \cos x) + C$$

EXERCISE 8.1

Evaluate the following :

1) $\int (4x^3 - 1)\; dx$

2) $\int \left(5x^4 + \sqrt{x} - \frac{7}{\sqrt{x}}\right) dx$

3) $\int \left(2x^3 + 8x + \frac{5}{x} + e^x\right) dx$

4) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

5) $\int \left(x + \frac{1}{x}\right)^3 dx$

6) $\int (5\sec x.\tan x + 2\operatorname{cosec}^2 x)\; dx$

7) $\int \left(\frac{x^{7/2} + x^{5/2} + 1}{x}\right) dx$

8) $\int \left(\frac{x^3 + 3x^2 + 4}{\sqrt{x}}\right) dx$

9) $\int \left(3e^{x} + \frac{2}{x\sqrt{x^{2}-1}} \right) dx$

10) $\int \left(\frac{x^{3}+1}{x^{4}} \right) dx$

11) $\int (3-2x)(2x+3)\ dx$

12) $\int \sqrt{x}\left(1+\sqrt{x}\right)^{2}\ dx$

13) $\int \left(\frac{1}{\sqrt[3]{x}} + 3\cos x - 7\sin x \right) dx$

14) $\int \frac{1-x}{\sqrt{x}}\ dx$

15) $\int \frac{x+2}{\sqrt{x+3}}\ dx$

16) $\int \frac{x+3}{\sqrt{x+1}}\ dx$

17) $\int \frac{x^{2}-1}{x^{2}+1}\ dx$

18) $\int \frac{x^{2}}{1+x^{2}}\ dx$

19) $\int \sqrt{1-\sin 2x}\ dx$

20) $\int \frac{dx}{1+\cos x}$

21) $\int \left(x^{-4} - e^{-x}\right)\ dx$

22) $\int \frac{e^{x}-x}{xe^{x}}\ dx$

23) $\int \left(x^{-1} - x^{-2} + e^{x}\right)\ dx$

24) $\int (3x+2)^{2}\ dx$

25) $\int \left(x^{-2} + e^{-2x} + 7\right)\ dx$

26) $\int \frac{1}{1-\sin x}\ dx$

8.2.1 Integration by substitution

Example 5

Evaluate $\int \frac{dx}{\sqrt{x}+x}$

Solution:

$$\sqrt{x}+x = \sqrt{x}\left(1+\sqrt{x}\right)$$

$$\text{Put } \left(1+\sqrt{x}\right) = t$$

$$\frac{1}{2\sqrt{x}}\ dx = dt$$

$$\therefore \int \frac{dx}{\sqrt{x}+x} = \int \frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)}$$

$$= \int \frac{2}{t} \quad dt$$

$$= 2 \log t + C \quad = 2 \log \left(1+\sqrt{x}\right) \quad + C$$

Example 6

Evaluate $\displaystyle\int \frac{1}{x^2} e^{-1/x} \quad dx$

Solution :

Put $\quad \dfrac{-1}{x} \quad = t$

$$\frac{1}{x^2} \quad dx = dt$$

$$\therefore \int \frac{1}{x^2} e^{-1/x} \quad dx = \int e^t \quad dt$$

$$= e^t + C$$

$$= e^{-1/x} \quad + \quad C$$

Example 7

Evaluate $\displaystyle\int \sec x \quad dx$

Solution :

$$\int \sec x \quad dx = \int \frac{\sec x \left(\sec x + \tan x\right)}{(\sec x + \tan x)} \quad dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\left(\sec x + \tan x\right)} \quad dx$$

Put $\sec x + \tan x = t$

$(\sec x \tan x + \sec^2 x)\, dx = dt$

$$\therefore \int \sec x \quad dx = \int \frac{dt}{t}$$

$$= \log t + C$$

Hence $\displaystyle\int \sec x \quad dx = \log(\sec x + \tan x) + C$

EXERCISE 8.2

Evaluate the following

1) $\int (2x-3)^{-5} \, dx$

2) $\int \frac{dx}{(3-2x)^2}$

3) $\int \sqrt[5]{4x+3} \, dx$

4) $\int e^{4x+3} \, dx$

5) $\int \frac{x^2}{(x-1)^{3/2}} \, dx$

6) $\int (3x^2+1)(x^3+x-4) \, dx$

7) $\int x \sin(x^2) \, dx$

8) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx$

9) $\int \frac{(\log x)^2}{x} \, dx$

10) $\int (2x+1)\sqrt{x^2+x} \, dx$

11) $\int \frac{x}{\sqrt{x^2+1}} \, dx$

12) $\int (x+1)(x^2+2x)^3 \, dx$

13) $\int \frac{2x+3}{x^2+3x+5} \, dx$

14) $\int \frac{x^2}{4+x^6} \, dx$

15) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

16) $\int \frac{dx}{x \log x}$

17) $\int \frac{\sec^2(\log x)}{x} \, dx$

18) $\int \frac{1}{(2x+1)^3} \, dx$

19) $\int \frac{dx}{x \log x \, \log(\log x)}$

20) $\int \frac{\sec^2 x}{(1-2\tan x)^4} \, dx$

21) $\int \cot x \, dx$

22) $\int \operatorname{cosec} x \, dx$

23) $\int \frac{dx}{x(1+\log x)}$

24) $\int \frac{x \tan^{-1} x^2}{1+x^4} \, dx$

25) $\int \frac{\sqrt{3+\log x}}{x} \, dx$

26) $\int \frac{dx}{x(x^4+1)}$

27) $\int \frac{\sec^2\sqrt{x} \tan\sqrt{x}}{\sqrt{x}} \, dx$

28) $\int \sqrt{2x+4} \, dx$

29) $\int (x^2-1)^4 \,.2x\,dx$

30) $\int (2x+1)\sqrt{x^2+x+4}\quad dx$

31) $\int \frac{\sec^2 x}{a+b\tan x}\quad dx$

32) $\int \tan x \quad dx$

8.2.2 Six Important Integrals

(i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + C$

(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + C$

(iv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(v) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right) + C$

(vi) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right) + C$

In the subsequent exercises let us study the application of the above formulae in evaluation of integrals.

Example 8

Evaluate $\int \frac{dx}{\sqrt{4-x^2}}$

Solution:

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{\sqrt{(2)^2-x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + c$$

Example 9

Evaluate $\int \frac{dx}{5+x^2}$

Solution:

$$\int \frac{dx}{5+x^2} = \int \frac{dx}{\left(\sqrt{5}\right)^2 + x^2}$$

$$= \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

Example 10

Evaluate $\int \frac{\mathbf{dx}}{\mathbf{x^2 - 7}}$

Solution:

$$\int \frac{dx}{x^2 - 7} = \int \frac{dx}{x^2 - \left(\sqrt{7}\right)^2}$$

$$= \frac{1}{2\sqrt{7}} \log\left(\frac{x-\sqrt{7}}{x+\sqrt{7}}\right) + C$$

Example 11

Evaluate $\int \frac{\mathbf{dx}}{\sqrt{\mathbf{4x^2 - 9}}}$

Solution:

$$\int \frac{dx}{\sqrt{4x^2 - 9}} = \int \frac{dx}{\sqrt{4\left(x^2 - 9/4\right)}}$$

$$= \frac{1}{2} \int \frac{dx}{\sqrt{x^2 - (3/2)^2}}$$

$$= \frac{1}{2} \log\left(x + \sqrt{x^2 - (3/2)^2}\right) + C$$

8.2.3 Integrals of the type $\int \frac{\mathbf{dx}}{\mathbf{ax^2 + bx + c}}$

If the denominator of the integrand is factorisable, then it can be split into partial fractions. Otherwise the denominator of the integrand can be written as the sum or difference of squares and then it can be integrated.

Example 12

Evaluate $\int \frac{dx}{7 + 6x - x^2}$

Solution:

$7 + 6x - x^2 = 7 - (x^2 - 6x)$
$= 7 - (x^2 - 6x + 9 - 9)$
$= 7 + 9 - (x - 3)^2$
$= 16 - (x-3)^2$

$$\therefore \int \frac{dx}{7 + 6x - x^2} = \int \frac{dx}{(4)^2 - (x-3)^2}$$

$$= \frac{1}{2 \times 4} \log\left(\frac{4 + (x-3)}{4 - (x-3)}\right) + C$$

$$= \frac{1}{8} \log\left(\frac{x+1}{7-x}\right) + C$$

Example 13

Evaluate $\int \frac{dx}{x^2 + 3x + 2}$

Solution:

$x^2 + 3x + 2 = (x + 1)(x + 2)$

Let $\frac{1}{x^2 + 3x + 2} = \frac{A}{x+1} + \frac{B}{x+2}$

$\Rightarrow \quad 1 = A(x+2) + B(x+1)$

When $x = -1$; $A = 1$

When $x = -2$; $B = -1$

$$\therefore \int \frac{dx}{x^2 + 3x + 2} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log(x+1) - \log(x+2) + C$$

$$= \log \frac{x+1}{x+2} + C$$

8.2.4 Integrals of the type $\int \frac{px + q}{ax^2 + bx + c}$ **dx** where $ax^2 + bx + c$ is not factorisable

To integrate a function of the form $\dfrac{px+q}{ax^2+bx+c}$, we write

$$px + q = A\,\frac{d}{dx}\left(ax^2+bx+c\right) + B$$

After finding the values of A and B we integrate the function, in usual manner.

Example 14

Evaluate $\displaystyle\int \frac{\mathbf{2x+7}}{\mathbf{2x^2+x+3}}\,\mathbf{dx}$

Solution:

Let $2x+7 = A\,\dfrac{d}{dx}(2x^2+x+3) + B$

$2x+7 = A(4x+1) + B$

Comparing the coefficient of like powers of x, we get

$4A = 2 \quad ; \quad A + B = 7$

$\Rightarrow \quad A = 1/2 \quad ; \quad B = 13/2$

$$\therefore \int \frac{2x+7}{2x^2+x+3}\,dx = \int \frac{1/2\,(4x+1) + 13/2}{2x^2+x+3}\,dx$$

$$= \frac{1}{2}\int \frac{4x+1}{2x^2+x+3}\,dx + \frac{13}{2}\int \frac{dx}{2x^2+x+3}$$

Let $I_1 = \dfrac{1}{2}\displaystyle\int \frac{4x+1}{2x^2+x+3}\,dx$ and $I_2 = \dfrac{13}{2}\displaystyle\int \frac{dx}{2x^2+x+3}$

$$I_1 = \frac{1}{2}\log\left(2x^2+x+3\right) + C_1$$

$$I_2 = \frac{13}{2}\int \frac{dx}{2x^2+x+3} = \frac{13}{4}\int \frac{dx}{(x+1/4)^2 + (3/2 - 1/16)}$$

$$= \frac{13}{4}\int \frac{dx}{(x+1/4)^2 + (\sqrt{23}/4)^2}$$

$$= \frac{13}{4}\times\frac{4}{\sqrt{23}}\tan^{-1}\left(\frac{x+1/4}{\sqrt{23}/4}\right) + C_2$$

$$\therefore \int \frac{2x+7}{2x^2+x+3}\,dx = \frac{1}{2}\log\left(2x^2+x+3\right) + \frac{13}{\sqrt{23}}\tan^{-1}\left(\frac{x+1/4}{\sqrt{23}/4}\right) + C$$

8.2.5 Integrals of the type $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

This type of integrals can be evaluated by expressing $ax^2 + bx + c$ as the sum or difference of squares.

Example 15

Evaluate $\int \frac{dx}{\sqrt{5 + 4x - x^2}}$

Solution:

$$\begin{aligned} 5 + 4x - x^2 &= -(x^2 - 4x - 5) \\ &= -(x^2 - 4x + 4 - 4 - 5) \\ &= -[(x-2)^2 - 9] \\ &= 9 - (x-2)^2 \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{5 + 4x - x^2}} &= \int \frac{dx}{\sqrt{9 - (x-2)^2}} \\ &= \int \frac{dx}{\sqrt{3^2 - (x-2)^2}} \\ &= \sin^{-1}\left(\frac{x-2}{3}\right) + C \end{aligned}$$

Example 16

Evaluate $\int \frac{dx}{\sqrt{4x^2 + 16x - 20}}$

Solution :

$$\begin{aligned} 4x^2 + 16 - 20 &= 4(x^2 + 4x - 5) \\ &= 4[x^2 + 4x + 4 - 4 - 5] \\ &= 4[(x+2)^2 - 9] \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4x^2 + 16x - 20}} &= \int \frac{dx}{\sqrt{4[(x+2)^2 - 9]}} \\ &= \frac{1}{2} \int \frac{dx}{\sqrt{(x+2)^2 - 3^2}} \\ &= \frac{1}{2} \log\left\{(x+2) + \sqrt{x^2 + 4x - 5}\right\} + C \end{aligned}$$

8.2.6 Integrals of the type $\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx$

To integrate such a function choose A and B such that

$$px + q = A \frac{d}{dx}(ax^2 + bx + c) + B$$

After finding the values of A and B we integrate the function in usual manner.

Example 17

Evaluate $\int \frac{2x + 1}{\sqrt{x^2 + 2x - 1}}\, dx$

Solution:

Let $2x+1 = A \frac{d}{dx}\left(x^2 + 2x - 1\right) + B$

$2x + 1 = A(2x + 2) + B$

Comparing Coefficients of like terms, we get

$2A = 2 \quad ; \quad 2A + B = 1$

$\Rightarrow \quad A = 1 \quad ; \quad B = -1$

$$\therefore \int \frac{2x+1}{\sqrt{x^2+2x-1}}\, dx = \int \frac{1.(2x+2)-1}{\sqrt{x^2+2x-1}}\, dx$$

$$= \int \frac{(2x+2)}{\sqrt{x^2+2x-1}}\, dx - \int \frac{dx}{\sqrt{x^2+2x-1}}$$

Let $I_1 = \int \frac{(2x+2)}{\sqrt{x^2+2x-1}}\, dx$

Put $x^2 + 2x - 1 = t^2$

$(2x + 2)\, dx = 2t\, dt$

$$\therefore I_1 = \int \frac{2t}{\sqrt{t^2}}\, dt = 2\int dt$$

$$= 2t$$

$$= 2\sqrt{x^2 + 2x - 1} + C_1$$

Let $I_2 = - \int \frac{dx}{\sqrt{x^2 + 2x - 1}}$

$$= - \int \frac{dx}{\sqrt{(x+1)^2 - (\sqrt{2})^2}} \quad = - \log\left((x+1) + \sqrt{x^2+2x-1}\right) + C_2$$

$$\therefore \quad \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx = 2\sqrt{x^2+2x-1} - \log\left((x+1)+\sqrt{x^2+2x-1}\right) + C$$

EXERCISE 8.3

Evaluate the following integrals

1) $\int \frac{1}{3+x^2} \, dx$

2) $\int \frac{dx}{2x^2+1}$

3) $\int \frac{dx}{x^2-4}$

4) $\int \frac{dx}{5-x^2}$

5) $\int \frac{dx}{\sqrt{9x^2-1}}$

6) $\int \frac{dx}{\sqrt{25+36x^2}}$

7) $\int \frac{dx}{\sqrt{9-4x^2}}$

8) $\int \frac{dx}{x^2+2x+3}$

9) $\int \frac{dx}{9x^2+6x+5}$

10) $\int \frac{dx}{\sqrt{x^2+4x+2}}$

11) $\int \frac{dx}{\sqrt{3-x+x^2}}$

12) $\int \frac{x+1}{x^2+4x-5} \, dx$

13) $\int \frac{7x-6}{x^2-3x+2} \, dx$

14) $\int \frac{x+2}{x^2-4x+3} \, dx$

15) $\int \frac{4x+1}{\sqrt{2x^2+x-3}} \, dx$

16) $\int \frac{2x+4}{\sqrt{x^2+2x-1}} \, dx$

8.2.7 Integration by parts

If u and v are functions of x such that u is differentiable and v is integrable, then

$$\int \mathbf{u} \; \mathbf{dv} = \mathbf{uv} - \int \mathbf{v} \; \mathbf{du}$$

Observation:

(i) When the integrand is a product, we try to simplify and use addition and subtraction rule. When this is not possible we use integration by parts.

(ii) While doing integration by parts we use 'ILATE' for the relative preference of u . Here,

I $\rightarrow$ Inverse trigonometic function

L $\rightarrow$ Logarithmic function

A $\rightarrow$ Algebraic function

T $\rightarrow$ Trigonometric function

E $\rightarrow$ Exponential function

Example 18

Evaluate $\int \mathbf{x.e^x}\ \mathbf{dx}$

Solution :

Let u = x , dv = e^x dx

du = dx , v = e^x

$$\int x.e^x \, dx = x e^x - \int e^x \, dx$$

$$= x e^x - e^x + C$$

$$= e^x (x - 1) + C$$

Example 19

Evaluate $\int \frac{\mathbf{\log x}}{(\mathbf{1+x})^2}\ \mathbf{dx}$

Solution:

Let u = log x ; dv = $\frac{dx}{(1+x)^2}$

du = $\frac{1}{x}$; v = $-\frac{1}{(1+x)}$

$$\int \frac{\log x}{(1+x)^2} dx = -(\log x)\left(\frac{1}{1+x}\right) - \int -\frac{1}{1+x}.\frac{1}{x} \, dx$$

$$= -\left(\frac{1}{1+x}\right)(\log x) + \int \frac{1}{x(1+x)} \, dx$$

$$= -\left(\frac{1}{1+x}\right)(\log x) + \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx$$

(Resolving into Partial Fractions)

$$= -\frac{1}{(1+x)}(\log x) + \log x - \log(1+x) + C$$

$$= -\frac{1}{(1+x)}(\log x) + \log\frac{x}{1+x} + C$$

Example 20

Evaluate $\int x.\sin 2x \; dx$

Solution:

Let u = x , sin2x dx = dv

du = dx , $\frac{-\cos 2x}{2} = v$

$$\int x.\sin 2x \; dx = \frac{-x\cos 2x}{2} + \int\frac{\cos 2x}{2} \; dx$$

$$= \frac{-x\cos 2x}{2} + \frac{1}{2}.\frac{\sin 2x}{2}$$

$$= \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} + C$$

Example 21

Evaluate $\int x^n \log x \; dx$, n ¹ -1

Solution:

Let u = log x , dv = x^n dx

du = $\frac{1}{x}$ dx , v = $\frac{x^{n+1}}{n+1}$

$$\int x^n \log x \; dx = \frac{x^{n+1}}{n+1}\log x - \int\frac{x^{n+1}}{n+1}\frac{1}{x} \; dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n \; dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\frac{x^{n+1}}{n+1} + C$$

$$= \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C$$

EXERCISE 8.4

Evaluate the following

1) $\int x\, e^{-x} \; dx$
2) $\int x \log x \; dx$
3) $\int \log x \; dx$
4) $\int x\, a^{x} \; dx$
5) $\int (\log x)^{2} \; dx$
6) $\int \frac{\log x}{x^{2}} \; dx$
7) $\int x \cos 2x \; dx$
8) $\int x \sin 3x \; dx$
9) $\int \cos^{-1} x \; dx$
10) $\int \tan^{-1} x \; dx$
11) $\int x \sec x \tan x \; dx$
12) $\int x^{2}\, e^{x} \; dx$

8.2.8 Standard Integrals

(i) $\int \sqrt{x^2 - a^2}\; dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + C$

(ii) $\int \sqrt{x^2 + a^2}\; dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + C$

(iii) $\int \sqrt{a^2 - x^2}\; dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

Example 22

Evaluate $\int \sqrt{\mathbf{49 - x^2}} \; \mathbf{dx}$

Solution :

$$\int \sqrt{49 - x^2}\; dx = \int \sqrt{(7)^2 - x^2}\; dx$$

$$= \frac{x}{2}\sqrt{49 - x^2} + \frac{49}{2}\sin^{-1}\left(\frac{x}{7}\right) + C$$

Example 23

Evaluate $\int \sqrt{\mathbf{16x^2 + 9}} \; \mathbf{dx}$

Solution :

$$\int \sqrt{16x^2 + 9}\; dx = \int \sqrt{16\left(x^2 + \frac{9}{16}\right)}\; dx$$

$$= 4 \int \sqrt{x^2 + \left(\frac{3}{4}\right)^2} \quad dx$$

$$= 4 \left\{ \frac{x}{2} \sqrt{x^2 + \left(\frac{3}{4}\right)^2} + \frac{\left(3/4\right)^2}{2} \log \left(x + \sqrt{x^2 + \left(\frac{3}{4}\right)^2} \right) \right\} + C$$

$$= \frac{x}{2} \sqrt{16x^2 + 9} + \frac{9}{8} \log \left(4x + \sqrt{16x^2 + 9}\right) + C$$

Example 24

Evaluate $\int \sqrt{\mathbf{x^2 - 16}} \quad \mathbf{dx}$

Solution :

$$\int \sqrt{x^2 - 16} \quad dx = \int \sqrt{x^2 - (4)^2} \quad dx$$

$$= \frac{x}{2} \sqrt{x^2 - 16} - \frac{16}{2} \log \left(x + \sqrt{x^2 - 16}\right) + C$$

$$= \frac{x}{2} \sqrt{x^2 - 16} - 8 \log \left(x + \sqrt{x^2 - 16}\right) + C$$

EXERCISE 8.5

Evaluate the following:

1) $\int \sqrt{x^2 - 36} \quad dx$ 2) $\int \sqrt{16 - x^2} \quad dx$

(3) $\int \sqrt{25 + x^2} \quad dx$ 4) $\int \sqrt{x^2 - 25} \quad dx$

5) $\int \sqrt{4x^2 - 5} \quad dx$ 6) $\int \sqrt{9x^2 - 16} \quad dx$

8.3 DEFINITE INTEGRAL

The definite integral of the continuous function f(x) between the limits x = a and x = b is defined as $\int_a^b f(x) \quad dx = \left[F(x) \right]_a^b$ = F (b) - F (a) where 'a' is the lower limit and 'b' is the upper limit and F(x) is the integral of f(x) .

To evaluate the definite integral, integrate the given function as usual . Then obtain the difference between the values by substituting the upper limit first and then the lower limit for x.

Example 25

Evaluate $\int_1^2 (4x^3 + 2x + 1) \, dx$

Solution:

$$\int_1^2 (4x^3 + 2x + 1) \, dx = \left[4\frac{x^4}{4} + 2\frac{x^2}{2} + x\right]_1^2$$

$= (2^4 + 2^2 + 2) - (1 + 1 + 1)$

$= (16 + 4 + 2) - 3$

$= 19$

Example 26

Evaluate $\int_2^3 \frac{2x}{1+x^2} \, dx$

Solution:

$$\int_2^3 \frac{2x}{1+x^2} \, dx = \int_5^{10} \frac{dt}{t}$$

Put $1 + x^2 = t$

$2x \, dx = dt$

When $x = 2$; $t = 5$

$x = 3$; $t = 10$

$= [\log t]_5^{10} = \log 10 - \log 5$

$= \log_e \frac{10}{5}$

$= \log_e 2$

Example 27

Evaluate $\int_1^{\sqrt{e}} x \log x \, dx$

Solution:

In $\int x \log x \, dx$

let $u = \log x \qquad dv = x\,dx$

$$du = \frac{1}{x}\,dx \qquad v = \frac{x^2}{2}$$

$$\int x \log x \quad dx = \frac{x^2}{2}\log x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x \quad dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\,\frac{x^2}{2}$$

$$\int_1^{\sqrt{e}} x \log x \quad dx = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_1^{\sqrt{e}}$$

$$= \left\{\frac{e}{2}\log\sqrt{e} - \frac{e}{4}\right\} - \left\{0 - \frac{1}{4}\right\}$$

$$= \frac{e}{2} \times \frac{1}{2} - \frac{e}{4} + \frac{1}{4}$$

$$= \frac{1}{4}$$

Example 28

Evaluate $\displaystyle \int_0^{\frac{\pi}{2}} \sin^2 x \quad dx$

[redacted]

$$\int \sin^2 x \quad dx = \int\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4}$$

$$\int_0^{\frac{p}{2}} \sin^2 x \quad dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{p}{2}}$$

$$= \frac{p}{4}$$

Example 29

Evaluate $\int_0^{\infty} \mathbf{xe^{-x^2}} \quad \mathbf{dx}$

Solution:

In $\int xe^{-x^2} \quad dx$

put $x^2 = t$

$2x\, dx = dt$

when $x = 0$; $t = 0$

$x = \infty$; $t = \infty$

$$\therefore \qquad \int_0^{\infty} xe^{-x^2} \quad dx = \int_0^{\infty} \frac{1}{2} e^{-t} \quad dt$$

$$= \frac{1}{2}\left[-e^{-t}\right]_0^{\infty}$$

$$= \frac{1}{2}[0 + 1]$$

$$= \frac{1}{2}$$

EXERCISE 8.6

Evaluate the following

1) $\int_1^2 \left(x^2 + x + 1\right) \quad dx$ 2) $\int_0^2 \frac{5}{2 + x}$ dx

3) $\int_0^1 \frac{dx}{1+x^2}$

4) $\int_0^1 2^x \quad dx$

5) $\int_0^3 e^{\frac{x}{3}} \quad dx$

6) $\int_0^1 xe^{x^2} \quad dx$

7) $\int_0^1 \frac{e^x}{1+e^{2x}} \quad dx$

8) $\int_0^{\frac{p}{4}} \tan^2 x \quad dx$

9) $\int_0^1 \frac{x}{1+x^4} \quad dx$

10) $\int_0^1 \frac{1-x^2}{1+x^2} \quad dx$

11) $\int_1^2 \log x \quad dx$

12) $\int_0^4 \sqrt{2x+4} \quad dx$

13) $\int_0^{\frac{p}{2}} \cos^2 x \quad dx$

14) $\int_0^{\frac{p}{2}} \frac{\cos x}{(1+\sin x)(2+\sin x)} \quad dx$

15) $\int_0^{\frac{p}{2}} \sqrt{1+\cos 2x} \quad dx$

16) $\int_1^{e^2} \frac{dx}{x(1+\log x)^2}$

17) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

18) $\int_0^1 x^3.e^{x^4} \quad dx$

8.3.1 Definite Integral as the Limit of the sum

Theorem:

Let the interval [a, b] be divided into n equal parts and let the width of each part be h, so that nh = b – a ; then

$$\int_a^b f(x) \quad dx \quad = \quad \underset{\substack{n\to\infty \\ h\to 0}}{Lt}\; h\left[f(a+h) + f(a+2h) + . \quad . \quad . \quad + f(a+nh)\right]$$

where a + h , a + 2h , a + 3h, . . . a + nh are the points of division obtained when the interval [a , b] is divided into n equal parts ; h being the width of each part.
[Proof is not required].

Example 30

Evaluate $\int_1^2 x^2 \, dx$ from the definition of an integral as the limit of a sum.

Solution :

$$\int_a^b f(x)dx = \underset{\substack{n\to\infty \\ h\to 0}}{Lt} h\left[f(a+h) + f(a+2h) + \ldots + f(a+nh)\right]$$

$$\int_a^b x^2 \quad dx = \underset{\substack{n\to\infty \\ h\to 0}}{Lt} h\left\{(a+h)^2 + (a+2h)^2 + \ldots + (a+nh)^2\right\}$$

$$= \underset{\substack{n\to\infty \\ h\to 0}}{Lt} h\left\{\left(a^2 + 2ah + h^2\right) + \left(a^2 + 4ah + 4h^2\right) + \ldots\left(a^2 + 2anh + n^2h^2\right)\right\}$$

$$= \underset{\substack{n\to\infty \\ h\to 0}}{Lt} h\left\{na^2 + 2ah\left(1 + 2 + 3 + \ldots + n\right) + h^2\left(1^2 + 2^2 + 3^2 + \ldots + n^2\right)\right\}$$

$$= \underset{\substack{n\to\infty \\ h\to 0}}{Lt} h\left\{na^2 + 2ah \frac{n(n+1)}{2} + \frac{h^2}{6} n \ (n+1) \ (2n+1)\right\}$$

$$\int_1^2 x^2 \ dx = \underset{n\to\infty}{Lt} \frac{1}{n}\left(n + \frac{1}{n} . n \ (n+1) + \frac{n(n+1) \ (2n+1)}{6n^2}\right)$$

$$= \underset{n\to\infty}{Lt} \left(1 + \frac{n+1}{n} + \frac{n(n+1) \ (2n+1)}{6n^3}\right)$$

$$= \underset{\frac{1}{n}\to 0}{Lt} \left(1 + 1 + \frac{1}{n} + \frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6n^3}\right)$$

$$= \underset{\frac{1}{n}\to 0}{Lt}\left(2+\frac{1}{n}+\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right)$$

$$= 2+\frac{2}{6} = \frac{7}{3}$$

EXERCISE 8.7

Evaluate the following definite integrals as limit of sums

1) $\int_1^2 x \, dx$ 2) $\int_0^1 e^x \, dx$

3) $\int_1^2 x^3 \, dx$ 4) $\int_0^1 x^2 \, dx$

EXERCISE 8.8

Choose the correct answer

1) The anti derivative of $= 5x^4$ is

(a) x^4 (b) x^5 (c) $4x^5 + c$ (d) $5x^4$

2) $\int 3 \, dx$ is

(a) 3 (b) x + C (c) 3x (d) 3x + c

3) $\int \frac{10}{x} \, dx$ is

(a) $\frac{1}{x}$ (b) $-\frac{1}{x^2}$ (c) 10 log x + C (d) log x + C

4) $\int e^{-x} \, dx$ is

(a) $-e^{-x} + C$ (b) $e^{-x} + C$ (c) $e^x + C$ (d) $-e^x + C$

5) $\int 21\sqrt{x} \, dx$ is

(a) $21x\sqrt{x}$ (b) $14x\sqrt{x} + C$ (c) $x\sqrt{x} + C$ (d) $\sqrt{x} + C$

6) $\int e^{5x} \quad dx$ is

(a) 5x + C (b) e^{5x} + C (c) $\frac{1}{5}$ e^{5x} + C (d) $\frac{1}{5}\ e^{5x}$

7) $\int \sin ax \quad dx$ is

(a) $\frac{-1}{a}\cos ax + C$ (b) $\frac{1}{a}\cos ax + C$ (c) sin ax + C (d) cos ax + C

8) $\int x^{-2} \quad dx$ is

(a) $\frac{1}{x} \quad + \quad C$ (b) $-\frac{1}{x} \quad + \quad C$ (c) $\frac{1}{x^2} + C$ (d) $-\frac{1}{x^2} + C$

9) $\int \frac{1}{2x} \quad dx$ is

(a) $\log \sqrt{x}$ + C (b) $\frac{1}{2}\log x$ (c) log x + C (d) $\frac{1}{\sqrt{2}}\log x$ + C

10) $\int e^{x+4} \quad dx$ is

(a) e^x + C (b) e^{x+4} + C (c) $\frac{e^{x+4}}{4}$ + C (d) e^{4x} + C

11) $\int 2\sec^2 x \quad dx$ is

(a) 2 tan x + C (b) $\sec^2 x$ tan x + C (c) $\tan^2 x$ + C (d) tan x + C

12) $\int 2^x .3^{-x} \quad dx$ is equal to

(a) $\frac{2}{3}\log x \quad + \quad C$ (b) $\frac{\left(2/3\right)^x}{\log_e 2/3}$ + C

(c) $\frac{\left(2/3\right)x}{\log_x 2/3}$ (d) $\log \left(\frac{2}{3}\right)^x$

13) $\int \frac{2}{x+1} \quad dx$ is equal to

(a) 2 log (x + 1) + C (b) 2 log (x + 1)+c
(c) 4 log (x + 1) + C (d) log (x + 1) + C

14) $\int (x+1)^8 \quad dx$ is equal to

(a) $\frac{(x+1)^9}{9} + C$ (b) $\frac{(x+1)^7}{7} + C$ (c) $(x+1)^8 + C$ (d) $(x+1)^4 + C$

15) $\int \frac{4x^3}{x^4+1} \quad dx$ is equal to

[redacted] [redacted]

(c) $\log(x^4+1) + C$ (d) None of these

16) $\int \operatorname{cosec} x \quad dx$ is equal to

(a) log (tan x/2) + C (b) log cosec x + C

(c) log tan x + C (d) log (cosec x + tanx)

17) $\int \frac{x^4}{1+x^5} \quad dx$ is equal to

(a) $\log(1+x^5)$ (b) $\log(1+x^4) + C$

(c) $\log(1+x^5) + C$ (d) $\frac{1}{5} \log(1+x^5) + C$

18) $\int \frac{dx}{x^2+a^2}$ is equal to

(a) $\tan^{-1}\frac{x}{a} + C$ (b) $\frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(c) $\tan^{-1}\frac{a}{x} + C$ (d) $\frac{1}{a}\sin^{-1}\frac{x}{a} + C$

19) $\int e^x [f(x) + f'(x)] \quad dx$ is equal to

[redacted] [redacted]

(c) $e^x + C$ (d) $e^{-x} + C$

20) $\int e^x (\sin x + \cos x) \quad dx$ is equal to

(a) $e^x \cos x + c$ (b) $e^x \sin x \cos x + C$

(c) $e^x + C \cos x$ (d) $e^x \sin x + C$

21) $\int \frac{dx}{1+4x^2}$ is equal to

(a) $\frac{1}{2}\tan^{-1} 2x + C$ (b) $\frac{1}{2}\tan^{-1} x + C$

(c) $\frac{1}{2}\tan^{-1}(x+c)$ (d) $\tan^{-1}(2x) + C$

22) $\int (2x+3)^3 \quad dx$ is equal to

(a) $\frac{(2x+3)^4}{4} + C$ (b) $\frac{(2x+3)^3}{8} + C$

(c) $\frac{(2x+3)^4}{8} + C$ (d) $\frac{(2x+3)^2}{16} + C$

23) The value of $\int_1^2 \frac{1}{x} \quad dx$ is

(a) log 2 (b) 0 (c) log 3 (d) 2 log 2

24) The value of $\int_{-1}^{1} x^2 \quad dx$ is

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $-\frac{2}{3}$ (d) $\frac{2}{3}$

25) The value of $\int_{-1}^{0} x^4$ dx is

(a) 0 (b) -1 (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

26) The value of $\int_0^1 (x^2+1)$ dx is

(a) $\frac{4}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $-\frac{4}{3}$

27) The value of $\int_0^1 \frac{x}{1+x^2} \quad dx$ is

(a) log 2 (b) 2 log 2 (c) log $\frac{1}{2}$ (d) log $\sqrt{2}$

28) The value of $\int_1^4 x\sqrt{x} \quad dx$ is

(a) $\frac{62}{5}$ (b) $\frac{32}{5}$ (c) $\frac{15}{4}$ (d) $\frac{31}{5}$

29) The value of $\int_0^{\frac{p}{3}} \tan x \quad dx$ is

(a) $\log \frac{1}{2}$ (b) $\log 2$ (c) $2 \log 2$ (d) $\log \sqrt{2}$

30) The value of $\int_0^{p} \sin x \quad dx$ is

(a) 1 (b) 0 (c) 2 (d) -2

31) The value of $\int_0^{\frac{p}{2}} \cos x \quad dx$ is

(a) 0 (b) 1 (c) -1 (d) 2

32) The value of $\int_{-\infty}^{0} \frac{e^x}{1+e^x} dx$ is

(a) 0 (b) 1 (c) $\frac{1}{2} \log 2$ (d) $\log 2$

33) The value of $\int_0^{\infty} e^{-x} \quad dx$ is

(a) 1 (b) 0 (c) ∞ (d) -1

34) The value of $\int_0^4 \frac{dx}{\sqrt{16 - x^2}}$ is

(a) $\frac{p}{4}$ (b) $\frac{p}{3}$ (c) $\frac{p}{6}$ (d) $\frac{p}{2}$

35) The value of $\int_{-1}^{1} \frac{dx}{1+x^2}$ is

(a) $\frac{p}{2}$ (b) $\frac{p}{4}$ (c) $-\frac{p}{4}$ (d) p

STOCKS, SHARES AND DEBENTURES 9

When the capital for a business is very large, a Joint Stock Company is floated to mobilize the capital. Those who take the initiative to start a joint stock company are called the promoters of the company. The company may raise funds for its requirements through the issue of stocks, shares and debentures. The value notified on their certificates is called ***Face Value*** or ***Nominal Value*** or ***Par Value***.

9.1 BASIC CONCEPTS

9.1.1 Shares

The total capital of a company may be divided into small units called ***shares***. For example, if the required capital of a company is Rs. 5,00,000 and is divided into 50,000 units of Rs. 10 each, each unit is called a share of face value Rs. 10. A share may be of any face value depending upon the capital required and the number of shares into which it is divided. The holders of the shares are called ***share holders***. The shares can be purchased or sold only in integral multiples.

9.1.2 Stocks

The shares may be fully paid or partly paid. A company may consolidate and convert a number of its fully paid up shares to form a single ***stock***. Stock being one lump amount can be purchased or sold even in fractional parts.

9.1.3. Debentures

The term ***Debenture*** is derived from the Latin word 'debere' which means 'to owe a debt'. A debenture is a loan borrowed by a company from the public with a guarantee to pay a certain percentage of interest at stated intervals and to repay the loan at the end of a fixed period.

9.1.4 Dividend

The profit of the company distributed among the share holders is called ***Dividend***. Each share holder gets dividend proportionate to the face value of the shares held. Dividend is usually expressed as a percentage.

9.1.5 Stock Exchange

Stocks, shares and debentures are traded in the Stock Exchanges (or Stock Markets). The price at which they are available there is called Market Value or Market Price. They are said to be quoted *at premium* or *at discount* or *at par* according as their market value is above or below or equal to their face value.

9.1.6 Yield or Return

Suppose a person invests Rs. 100 in the stock market for the purchase of a stock. The consequent annual income he gets from the company is called ***yield*** or return. It is usually expressed as a percentage.

9.1.7 Brokerage

The purchase or sale of stocks, shares and debentures is done through agents called Stock Brokers. The charge for their service is called ***brokerage***. It is based on the face value and is usually expressed as a percentage. Both the buyer and seller pay the brokerage.

When stock is purchased, brokerage is added to cost price. When stock is sold, brokerage is subtracted from the selling price.

9.1.8 Types of Shares

There are essentially two types of shares

(i) Preference shares

(ii) Equity shares (ordinary shares)

Preference share holders have the following preferential rights

(i) The right to get a fixed rate of dividend before the payment of dividend to the equity holders.

(ii) The right to get back their capital before the equity holders in case of winding up of the company.

9.1.9 Technical Brevity of Quotation

By a '15% stock at 120' we mean a stock of face value Rs. 100, market value Rs. 120 and dividend 15%

9.1.10 Distinction Between Shares and Debentures

The following are the main differences between shares and debentures.

SHARES	DEBENTURES
1. Share money forms a part of the capital of the comapny. The share holders are part proprietors of the comapny.	1. Debentures are mere debts. Debenture holders are just creditors.
2. Share holders get dividend only out of profits and in case of insufficient or no profits they get nothing.	2. Debenture holders being creditors get guaranteed interest, as agreed, whether the company makes profit or not.
3. Share holders are paid after the debenture holders are paid their due first.	3. Debenture holders have to be paid first their interest due.
4. The dividend on shares depends upon the profit of the company.	4. The interest on debentures is very well fixed at the time of issue itself.
5. Shares are not to be paid back by the company	5. Debentures have to be paid back at the end of a fixed period.
6. In case the company is wound up, the share holders may lose a part or full of their capital	6. The debenture holders invariably get back their investment.
7. Investment in shares is speculative and has an element of risk associated with it.	7. The risk is very minimal.
8. Share holders have a right to attend and vote at the meetings of the share holders.	8. Debenture holders have no such rights.

We shall now take up the study of the mathematical aspects concerning the purchase and sale of stocks, shares and debentures by the following examples.

Example 1

Find the yearly income on 120 shares of 7% stock of face value Rs. 100

Solution:

Face Value (Rs.)	Yearly income (Rs.)
100	7
120 x 100	?

$$\text{Yearly income} = \frac{120 \times 100}{100} \times 7$$

$$= \text{Rs. } 840$$

Example 2

Find the amount of 8% stock that will give an annual income of Rs. 80.

Solution:

Income (Rs.)	Stock (Rs.)
8	100
80	?

$$\text{Stock} = \frac{80}{8} \times 100$$

$$= \text{Rs. } 1{,}000$$

Example 3

Find the number of shares which will give an annual income of Rs. 360 from 6% stock of face value Rs. 100.

Solution:

Income (Rs.)	Stock (Rs.)
6	100
360	?

Stock = $\frac{360}{6}$ x 100 = Rs. 6,000

∴ Number of shares = $\frac{6000}{100}$ = 60.

Example 4

Find the rate of dividend which gives an annual income of Rs. 1,200 for 150 shares of face value Rs. 100.

Solution:

Stock (Rs.)	Income (Rs.)
150 x 100	1200
100	?

Income = $\frac{100}{150 \times 100}$ x 1200

= Rs. 8

Rate of dividend = 8%

Example 5

Find how much 7% stock at 70 can be bought for Rs. 8,400.

Solution:

Investment (Rs.)	Stock (Rs.)
70	100
8400	?

Stock = $\frac{8,400}{70}$ x 100

= Rs. 12,000

Example 6

A person buys a stock for Rs. 9,000 at 10% discount. If the rate of dividend is 20% find his income.

Solution:

Investment (Rs.)	Income (Rs.)
90	20
9000	?

Income $= \frac{9,000}{90} \times 20$

$= \text{Rs. } 2,000$

Example 7

Find the purchase price of Rs. 9,300, 8 $\frac{3}{4}$ % stock at 4% discount.

Solution:

Stock (Rs.)	Purcahse Price (Rs.)
100	(100-4) = 96
9300	?

Purchase Price $= \frac{9,300}{100} \times 96$

$= \text{Rs. } 8,928$

Example 8

What should be the price of a 9% stock if money is worth 8%

Solution:

Income (Rs.)	Purchase Price (Rs.)
8	100
9	?

Purchase Price $= \frac{9}{8} \times 100$

$= \text{Rs. } 112.50$

Example 9

Sharala bought shares of face value Rs.100 of a 6% stock for Rs. 7,200. If she got an income of Rs. 540, find the purchase value of each share of the stock.

Solution:

Income (Rs.)	Purchase Price (Rs.)
540	7200
6	?

Purchase Price $= \frac{6}{540} \times 7200$

$= \text{Rs. } 80$

Example 10

Find the yield on 20% stock at 80.

Solution:

Investment (Rs.)	Income (Rs.)
80	20
100	?

Yield $= \frac{100}{80} \times 20$

$= 25\%$

Example 11

Find the yield on 20% stock at 25% discount.

Solution:

Investment (Rs.)	Income (Rs.)
(100-25) = 75	20
100	?

Yield $= \frac{100}{75} \times 20$

$= 26\frac{2}{3}\ \%$

Example 12

Find the yield on 20% stock at 20% premium.

Solution:

Investment (Rs.)	Income (Rs.)
120	20
100	?

Yield $= \frac{100}{120} \times 20$

$= 16\frac{2}{3}\ \%$

Example 13

Find the yield on 10% stock of face value Rs. 15 quoted at Rs. 10

Solution:

Investment (Rs.)	Face value (Rs.)
10	15
100	?

Face Value $= \frac{100}{10} \times 15$

$= \text{Rs. } 150$

Now,

Face value (Rs.)	Income (Rs.)
100	10
150	?

Yield $= \frac{150}{100} \times 10$

$= 15\%$

Example 14

Which is better investment : 7% stock at 80 or 9% stock at 96?

Solution:

Consider an imaginary investment of Rs. (80 x 96) in each stock.

<u>7% Stock</u>

Investment (Rs.)	Income (Rs.)
80	7
80 x 96	?

Income $= \frac{80 \times 96}{80} \times 7$

$= \text{Rs. } 672$

<u>9% Stock</u>

Investment (Rs.)	Income (Rs.)
96	9
80 x 96	?

Income $= \frac{80 \times 96}{96} \times 9$

$= \text{Rs. } 720$

For the same investment, 9% stock fetches more annual income than 7% stock.

∴ 9% stock at 96 is better.

Example 15

Which is better investment : 20% stock at 140 or 10% stock at 70?

Solution:

Consider an imaginary investment of Rs. (140 x 70) in each stock.

20% Stock

Investment (Rs.)	Income (Rs.)
140	20
140 x 70	?

$$\text{Income} = \frac{140 \times 70}{140} \times 20$$

= Rs. 1,400

10% Stock

Investment (Rs.)	Income (Rs.)
70	10
140 x 70	?

$$\text{Income} = \frac{140 \times 70}{70} \times 10$$

= Rs. 1,400

For the same investment, both stocks fetch the same income

∴ They are equivalent stocks.

Example 16

A man bought 6% stock of Rs. 12,000 at 92 and sold it when the price rose to 96. Find his gain.

Solution:

Stock (Rs.)	Gain (Rs.)
100	(96-92) = 4
12000	?

$$\text{Gain} = \frac{12000}{100} \times 4$$

= Rs. 480

Example 17

How much would a person lose by selling Rs. 4,250 stock at 87 if he had bought it at 105?

Solution:

Stock (Rs.)	Loss (Rs.)
100	(105-87) = 18
4250	?

Loss $= \frac{4250}{100} \times 18$

$= \text{Rs. } 765$

Example 18

Find the brokerage paid by Ram on his sale of Rs. 400 shares of face value Rs. 25 at $\frac{1}{2}$ % brokerage.

Solution:

Face Value (Rs.)	Brokerage (Rs.)
100	$\frac{1}{2}$
400 x 25	?

Brokerage $= \frac{400 \times 25}{100} \times \frac{1}{2}$

$= \text{Rs. } 50$

Example 19

Shiva paid Rs. 105 to a broker for buying 70 shares of face value Rs. 100. Find the rate of brokerage.

Solution:

Face Value (Rs.)	Brokerage (Rs.)
70 x 100	105
100	?

Rate of Brokerage $= \frac{100}{70 \times 100} \times 105$

$= 1\frac{1}{2}$ %

Example 20

A person buys a stock of face value Rs. 5,000 at a discount of 9 $\frac{1}{2}$ %, paying brokerage at $\frac{1}{2}$ %. Find the purchase price of the stock.

Solution:

Face Value (Rs.)	Purchase Price (Rs.)
100	$(100-9\frac{1}{2} + \frac{1}{2}) = 91$
5000	?

Purchase Price $= \frac{5000}{100} \times 91$

= Rs. 4,550

Example 21

A person sells a stock at a premium of 44%. The brokerage paid is 2%. If the face value of the stock is Rs. 20,000, what is the sale proceeds?

Solution:

Face Value (Rs.)	Sale Proceeds (Rs.)
100	(100+44-2) = 142
20,000	?

Sale Proceeds $= \frac{20000}{100} \times 142$

= Rs. 28,400

Example 22

A person buys a 15% stock for Rs. 7,500 at a premium of 18%. Find the face value of the stock purchased and the dividend, brokerage being 2%.

Solution:

Purchase Price (Rs.)	Face Value (Rs.)
(100+18+2) = 120	100
7,500	?

Face Value $= \frac{7500}{120} \times 100$

= Rs. 6,250

Also

Face value (Rs.)	Dividend (Rs.)
100	15
6,250	?

$$\text{Dividend} = \frac{6250}{100} \times 15$$

$$= \text{Rs. } 937.50$$

Example 23

Ram bought a 9% stock for Rs. 5,400 at a discount of 11%. If he paid 1% brokerage, find the percentage of his income.

Solution:

Investment (Rs.)	Income (Rs.)
(100-11+1) = 90	9
100	?

$$\text{Income} = \frac{100}{90} \times 9$$

$$= 10\%$$

Example 24

Find the investment requierd to get an income of Rs. 1938 from $9\frac{1}{2}$ % stock at 90. (Brokerage 1%)

Solution:

[redacted]	[redacted]
$9\frac{1}{2}$	(90+1) = 91
1938	?

$$\text{Investment} = \frac{1938}{9\frac{1}{2}} \times 91$$

$$= \frac{1938}{\frac{19}{2}} \times 91$$

$$= 1938 \times \frac{2}{19} \times 91$$

$$= \text{Rs. } 18,564$$

Example 25

Kamal sold Rs. 9,000 worth 7% stock at 80 and invested the proceeds in 15% stock at 120. Find the change in his income.

Solution:

<u>7% Stock</u>

Stock (Rs.)	Income (Rs.)
100	7
9000	?

$$\text{Income} = \frac{9000}{100} \times 7$$

= Rs. 630 ----- (1)

Also

Stock (Rs.)	Sale Proceeds (Rs.)
100	80
9000	?

$$\text{Sale Proceeds} = \frac{9000}{100} \times 80$$

= Rs. 7,200

<u>15% Srock</u>

Investment (Rs.)	Income (Rs.)
120	15
7,200	?

$$\text{Income} = \frac{7200}{120} \times 15$$

= Rs. 900 ------------ (2)

comparing (1) and (2), we conclude that the change in income (increase).

= Rs. 270

Example 26

A person sells a 20% stock of face value Rs. 5,000 at a premium of 62%. With the money obtained he buys a 15% stock at a discount of 22% What is the change in his income. (Brokerage 2%)

Solution:

<u>20% Stock</u>

Face Value (Rs.)	Income (Rs.)
100	20
5,000	?

Income $= \frac{5000}{100} \times 20$

= Rs. 1,000 -------------- (1)

Also,

Face Value (Rs.)	Sale Proceeds (Rs.)
100	(162-2) = 160
5,000	?

Sale Proceeds $= \frac{5000}{100} \times 160$

Rs. 8,000

<u>15% Stock</u>

Investment (Rs.)	Income (Rs.)
(100-22+2)= 80	15
8,000	?

Income $= \frac{8000}{80} \times 15$

Rs. 1,500 ------------ (2)

comapring (1) and (2) we conclude that the change in income (increase) = Rs. 500.

Example 27

Equal amounts are invested in 12% stock at 89 and 8% stock at 95 (1% brokerage paid in both transactions). If 12% stock brought Rs. 120 more by way of dividend income than the other, find the amount invested in each stock.

Solution:

Let the amount invested in each stock be Rs. x

<u>12% Stock</u>

Investment (Rs.)	Income (Rs.)
(89+1) = 90	12
x	?

Income $= \frac{x}{90} \times 12$

$= \text{Rs. } \frac{2x}{15}$

<u>8% Stock</u>

Investment (Rs.)	Income (Rs.)
(95+1) = 96	8
x	?

Income $= \frac{x}{96} \times 8$

$= \text{Rs. } \frac{x}{12}$

As per the problem,

$\frac{2x}{15} - \frac{x}{12} = 120$

Multiply by the LCM of 15 and 12 ie. 60

ie. $8x - 5x = 7200$

ie. $3x = 7200$

ie. $x = \text{Rs. } 2{,}400$

Example 28

Mrs. Prema sold Rs. 8,000 worth, 7% stock at 96 and invested the amount realised in the shares of face value Rs. 100 of a 10% stock by which her income increased by Rs. 80. Find the purchase price of 10% stock.

Solution:

<u>7% Stock</u>

Stock (Rs.)	Income (Rs.)
100	7
8,000	?

Income $= \frac{8000}{100} \times 7$

= Rs. 560

Also

Stock (Rs.)	Sale Proceeds (Rs.)
100	96
8,000	?

Sale proceeds $= \frac{8000}{100} \times 96$

= Rs. 7,680

<u>10% Stock</u>

Income = Rs. (560 + 80) = Rs. 640.

Income (Rs.)	Purchase Price (Rs.)
640	7680
10	?

Purchase Price $= \frac{10}{640} \times 7680$

= Rs. 120

Example 29

A company has a total capital of Rs. 5,00,000 divided into 1000 preference shares of 6% dividend with par value of Rs. 100 each and 4,000 ordinary shares of par value of Rs. 100 each. The company declares an annual dividend of Rs. 40,000. Find the dividend received by Mr. Gopal having 100 preference shares and 200 ordinary shares.

Solution :

Preference Shares = Rs. (1,000 x 100)

= Rs. 1,00,000

Ordinary Shares = Rs. (4,000 x 100)

= Rs. 4,00,000

Total dividend = Rs. 40,000

Dividend to preference shares

Shares (Rs.)	Dividend (Rs.)
100	6
1,00,000	?

Dividend = Rs. 6,000

Dividend to ordinary shares

= Rs. (40,000 - 6,000)

= Rs. 34,000

Gopal's Income from preference shares

Share (Rs.)	Dividend (Rs.)
1,00,000	6,000
100 x 100	?

Dividend $= \frac{100 \times 100}{100000} \times 6,000$

= Rs. 600

Gopal's income from ordinary shares

Share (Rs.)	Dividend (Rs.)
4,00,000	34,000
200 x 100	?

Dividend $= \frac{200 \times 100}{400000} \times 34,000$

= Rs. 1,700

Total Income received by Gopal

= Rs. (600 + 1700)

= Rs. 2,300

Example 30

The capital of a company is made up of 50,000 preference shares with a dividend of 16% and 25,000 ordinary shares. The par value of each of preference and ordinary shares is Rs. 10. The company had a total profit of Rs. 1,60,000. If Rs. 20,000 were kept in reserve and Rs. 10,000 in depreciation fund, what percent of dividend is paid to the ordinary share holders.

Solution:

Preference Shares = Rs. (50000 x 10)

= Rs. 5,00,000

Ordinary Shares = Rs. (25,000 x 10)

= Rs. 2,50,000

Total dividend = Rs. (1,60,000 - 20,000 - 10,000)

= Rs. 1,30,000

<u>Dividend to Preference Shares</u>

Shares (Rs.)	Dividend (Rs.)
100	16
5,00,000	?

Dividend $= \frac{500000}{100} \times 16$

= Rs. 80,000

<u>Dividend to ordinary shares</u>

= Rs. (1,30,000 - 80,000)

= Rs. 50,000

Now for ordinary shares,

Share (Rs.)	Dividend (Rs.)
2,50,000	50,000
100	?

Dividend $= \frac{100}{250000} \times 50{,}000$

= 20%

9.2 EFFECTIVE RATE OF RETURN ON DEBENTURES WITH NOMINAL RATE

When the interest for a debenture is paid more than once in a year the debenture is said to have a nominal rate. We can find the corresponding effective rate using the formula.

$$E = \frac{F}{M}\left[\left(1+\frac{i}{k}\right)^{k} - 1\right]$$

where

E	=	Effective rate of return
F	=	Face value of the debenture
M	=	Corresponding market value of the debenture
i	=	nominal rate on unit sum per year
k	=	the number of times the nominal rate is paid in a year.

Example 31

Find the effective rate of return on 15% debentures of face value Rs. 100 issued at a premium of 2% interest being paid quarterly.

Solution :

$$E = \frac{F}{M}\left[\left(1+\frac{i}{k}\right)^{k}-1\right]$$

$$= \frac{100}{102}\left[\left(1+\frac{0.15}{4}\right)^{4}-1\right]$$

$$= \frac{100}{102}\left[(1+0.0375)^{4}-1\right]$$

$$= \frac{100}{102}\left[(1.0375)^{4}-1\right]$$

$$= \frac{100}{102}[1.160-1]$$

$$= \frac{100}{102}[0.160]$$

$$= 0.1569 = 15.69\%$$

Logarithmic Calculation

log 1.0375 = 0.0161
4 x

0.0644
antilog 0.0644
= 1.160

log 100 = 2.0000
log 0.160 = $\bar{1}$.2041 +

1.2041
log 102 = 2.0086 -

$\bar{1}$.1955
antilog $\bar{1}$.1955
= 0.1569

Example 32

Find the effective rate of return on 16% Water Board bonds of face value Rs. 1,000 offered at Rs. 990, interest being paid half yearly.

Solution :

$$E = \frac{F}{M}\left[\left(1+\frac{i}{k}\right)^{k}-1\right]$$

$$= \frac{1000}{990}\left[\left(1+\frac{0.16}{2}\right)^{2}-1\right]$$

$= \frac{100}{99}\left[(1+0.08)^2 - 1\right]$

$= \frac{100}{99}\left[(1.08)^2 - 1\right]$

$= \frac{100}{99}[1.166]$

$= \frac{100}{99}[0.166]$

= 0.1677

= 16.77 %

Logarithmic Calculation

log 1.08	=	0.0334
		2 x

		0.0668
antilog		0.0668
	=	1.166

log 100	=	2.0000
log 0.166	=	$\bar{1}.2201$ +

		1.2201
log 99	=	1.9956 -

		$\bar{1}.2245$

antilog $\bar{1}.2245$

= 0.1677

EXERCISE 9.1

1) Find the yearly income on 300 shares of 10% stock of face value Rs. 25.

2) Find the amount of 9% stock which will give an annual income of Rs. 90.

3) Find the number of shares which will give an annual income of Rs. 900 from 9% stock of face value of Rs. 100.

4) Find how much of a 9% stock can be bought for Rs. 6,480 at 90.

5) Determine the annual income realised by investing Rs. 22,400 a t $7\frac{1}{2}$ % stock at 112.

6) Find the purchase price of a Rs. 9,000, 8% stock at 4% premium.

7) Find the percentage income on an investment in 8% stock at 120.

8) Krishna invested in 12% stock at 80. Find the rate of return.

9) Find the yield on 15% stock at 120.

10) Find the yield on 18% stock at 10% discount.

11) Find the yield on 8% stock at 4% premium.

12) Which is better investment, 6% stock at 120 or 5% stock at 95?

13) Which is better investment, 18% debentures at 10% premium or 12% debentures at 4% discount?

14) Find the yield on 12% debenture of face value Rs. 70 quoted at a discount of 10%

15) How much money should a person invest in 18%, Rs. 100 debentures available at 90 to earn an income of Rs. 8,100 annually.

16) A person bought shares of face value Rs. 100 of 10% stock by investing Rs. 8,000 in the market. He gets an income of Rs. 500. Find the purchase price of each share bought.

17) Mr. Sharma bought a 5% stock for Rs. 3,900. If he gets an annual income of Rs150, find the purchase price of the stock.

18) How much would a person lose by selling Rs. 4,500 stock at 90 if he had bought it for 105.

19) Find the brokerage paid by Mr. Ganesh on his sale of 350 shares of face value Rs. 100 at $1\frac{1}{2}$ % brokerage.

20) Mr. Ramesh bought 500 shares of par value Rs. 10. If he paid Rs. 100 as brokerage, find the rate of brokerage.

21) How much of 8% stock at a premium of 9% can be purchased with Rs. 6050 if brokerage is 1%

22) A person buys a 10% stock for Rs.1035 at a premium of 14%. Find the face value and the dividend, brokerage being 1%.

23) Mr. James sells 20% stock of face value Rs. 10,000 at 102. With the proceeds he buys a 15% stock at 12% discount. Find the change in his income. (Brokerage being 2%)

24) Mrs. Kamini sold Rs. 9,000 worth 7% stock at 80 and invested the sale proceeds in 15% stock by which her income increased by Rs. 270. Find the purchase price of 15% stock.

25) Mr. Bhaskar invests Rs. 34,000 partly in 8% stock at 80 and the remaining in $7\frac{1}{2}$ % stock at 90. If his annual income be Rs. 3,000, how much stock of each kind does he hold?

26) A company's total capital of Rs. 3,00,000 consists of 1000 preferential shares of 10% stock and remaining equity stock. In a year the company decided to distribute Rs. 20,000 as dividend. Find the rate of dividend for equity stock if all the shares have a face value of Rs. 100.

27) A 16% debenture is issued at a discount of 5%. If the interest is paid half yearly, find the effective rate of return.

EXERCISE 9.2

Choose the correct answer

1) A stock of face value 100 is traded at a premium. Then its market price may be

(a) 90 (b) 120 (c) 100 (d)none of these

2) A share of face value 100 is traded at 110. If 1% brokerage is to be paid then the purchase price of the share is

(a) 109 (b) 111 (c) 100 (d)none of these

3) A share of face value 100 is traded at 110. If 1% brokerage is to be paid then the sale proceeds of the share is

(a) 109 (b) 111 (c) 100 (d)none of these

4) The calculation of dividend is based on

(a) Face value (b) Market Value (c) Capital (d)none of these

5) Rs. 8,100 is invested to purchase a stock at 108. The amount of stock purchased is.

(a) Rs. 7,500 (b) Rs. 7,000 (c) Rs. 7,300 (d) Rs. 7,800

6) The investment required to buy a stock of Rs. 5,000 at 102 is

(a) Rs. 6,000 (b) Rs. 5,300 (c) Rs. 5,200 (d) Rs. 5,100

7) The sale proceeds on the sale of a stock of Rs. 10,000 at a permium of 10% is

(a) Rs. 12,000 (b) Rs. 11,000

(c) Rs. 6,000 (d) Rs. 12,500

8) The yield on 9% stock at 90 is

(a) 10% (b) 9% (c) 6% (d) 8%

9) The yield on 14% debenture of face value Rs. 200 quoted at par is

(a) 14% (b) 15% (c) 7% (d) 28%

10) By investing Rs. 8,000 in the Stock Market for the purchase of the shares of face value Rs. 100 of a company, Mr. Ram gets an income of Rs. 200, the dividend being 10%. Then the market value of each share is

(a) Rs. 280 (b) Rs. 250 (c) Rs. 260 (d) Rs. 400

11) The yield from 9% stock at 90 is

(a) 6% (b) 10% (c) 6.75% (d) 6.5%

12) If 3% stock yields 4%, then the market price of the stock is

(a) Rs. 75 (b) Rs. 133 (c) Rs. 80 (d) Rs. 120

STATISTICS 10

10.1 MEASURES OF CENTRAL TENDENCY

"An average is a value which is typical or representative of a set of data" - Murray R.Speiegel

Measures of central tendency which are also known as averages, gives a single value which represents the entire set of data. The set of data may have equal or unequal values.

Measures of central tendency are also known as "Measures of Location".

It is generally observed that the observations (data) on a variable tend to cluster around some central value. For example, in the data on heights (in cms) of students, majority of the values may be around 160 cm. This tendency of clustering around some central value is called as central tendency. A measure of central tendency tries to estimate this central value.

Various measures of Averages are

(i) Arithmetic Mean
(ii) Median
(iii) Mode
(iv) Geometric Mean
(v) Harmonic Mean

Averages are important in statistics Dr.A.L.Bowley highlighted the importance of averages in statistics as saying "Statistics may rightly be called the Science of Averages".

Recall : Raw Data

For individual observations $x_1, x_2, \ldots x_n$

(i) Mean $= \overline{x} = \frac{\Sigma x}{n}$

(ii) Median = Middle value if ‘n’ is odd

= Average of the two middle values if ‘n’ even

(iii) Mode = Most frequent value

Example 1

Find Mean, Median and Mode for the following data

3, 6, 7, 6, 2, 3, 5, 7, 6, 1, 6, 4, 10, 6

Solution:

Mean = $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{3+6+7+....+4+10+6}{14} = 5.14$

Median :

Arrange the above values in ascending (descending) order

1, 2, 3, 3, 4, 5, 6, 6, 6, 6, 6, 7, 7, 10

Here n = 14, which is even

$\therefore$ Median = Average two Middle values

= 6

Mode = 6 ($\because$ the values 6 occur five times in the above set of observation)

Grouped data (discrete)

For the set of values (observation) $x_1, x_2, ... x_n$ with corresponding frequences $f_1, f_2, ...f_n$

(i) Mean = $\bar{x} = \frac{\Sigma fx}{N}$, where N = Σf

(ii) Median = the value of x, corresponding to the cumulative frequency just greater than $\frac{N}{2}$

(iii) Mode = the value of x, corresponding to a maximum frequency.

Example 2

Obtain Mean, Median, Mode for the following data

Value (x)	**0**	**1**	**2**	**3**	**4**	**5**
Frequency (f)	**8**	**10**	**11**	**15**	**21**	**25**

Solution:

x	0	1	2	3	4	5
f	8	10	11	15	21	25
fx	0	10	22	48	80	125
cf	8	18	29	44	65	90

$N = \Sigma f = 90$

$\Sigma fx = 285$

$\therefore$ Mean $= \frac{\Sigma fx}{N}$

$= 3.17$

Median :

$N = \Sigma f = 90$

$\frac{N}{2} = \frac{90}{2} = 45$

the cumulative frequency just greater than $\frac{N}{2} = 45$ is 65.

$\therefore$ The value of x corresponding to c.f. 65 is 4.

$\therefore$ Median = 4

Mode :

Here the maximum frequency is 25. The value of x, which corresponding to the maximum frequency (25) is 5.

$\therefore$ Mode = 5

10.1.1 Arithmetic Mean for a continuous distribution

The formula to calculate arithmetic mean under this type is

$$\overline{X} = A + \left(\frac{\Sigma fd}{N} \times C\right)$$

where A = arbitrary value (may or may not chosen from the mid points of class-intervals.

d = $\frac{x-A}{c}$ is deviations of each mid values.

c = magnitude or length of the class interval.

N = Σf = total frequency

Example 3

Calculate Arithmetic mean for the following

Marks	**20-30**	**30-40**	**40-50**	**50-60**	**60-70**	**70-80**
No.of Students	**5**	**8**	**12**	**15**	**6**	**4**

Solution:

Marks	No. of Students	Mid value x	$d = \frac{x-A}{c}$ A=55, c=10	fd
20-30	5	25	-3	-15
30-40	8	35	-2	-16
40-50	12	45	-1	-12
50-60	15	55	0	0
60-70	6	65	1	6
70-80	4	75	2	8
	N = Sf = 50			**Sfd = -29**

∴ Arithmetic mean,

$$\overline{x} = A + \left(\frac{\Sigma fd}{N} \times c\right)$$

$$= 55 + \left(\frac{-29}{50} x 10\right) = 49.2$$

Example 4

Calculate the Arithmetic mean for the following

Wages in Rs. :	**100-119**	**120-139**	**140-159**	**160-179**	**180-199**
No.of Workers :	**18**	**21**	**13**	**5**	**3**

Solution:

Wages	No. of workers f	Mid value x	$d = \frac{x-A}{c}$ A=149.5, c=20	fd
100-119	18	109.5	-2	-36
120-139	21	129.5	-1	-21
140-159	13	149.5	0	0
160-179	5	169.5	1	5
180-199	3	189.5	2	6
	N = Sf = 60			**Sfd = -46**

$$\bar{x} = A + \left(\frac{\Sigma fd}{N} \times c\right)$$

$$= 149.5 + \left(\frac{-46}{60} \times 20\right) = 134.17$$

10.1.2 Median for continuous frequency distribution

In case of continuous frequency distribution, Median is obtained by the following formula.

$$\text{Median} = l + \left(\frac{\frac{N}{2} - m}{f} \times c\right)$$

where l = lower limit of the Median class.

m = c.f. of the preceding (previous) Median class

f = frequency of the Median class

c = magnitude or length of the class interval corresponding to Median class.

N = Σf = total frequency.

Example 5

Find the Median wage of the following distribution

Wages (in Rs.) :	**20-30**	**30-40**	**40-50**	**50-60**	**60-70**
No.of labourers:	**3**	**5**	**20**	**10**	**5**

Solution :

Wages	No. of labourers f	Cumulative frequency c.f.
20-30	3	3
30-40	5	8
40-50	20	28
50-60	10	38
60-70	5	43
	N = Sf = 43	

Here $\frac{N}{2} = \frac{43}{2} = 21.5$

cumulative frequency just greater than 21.5 is 28 and the corresponding median class is 40-50

$\Rightarrow l = 40$, m = 8, f = 20, c = 10

$$\therefore \text{Median} = l + \left(\frac{\frac{N}{2} - m}{f} \times c\right)$$

$$= 40 + \left(\frac{21.5-8}{20} \times 10\right) = \text{Rs. } 46.75$$

Example 6

Calculate the Median weight of persons in an office from the following data.

Weight (in kgs.)	**:**	**60-62**	**63-65**	**66-68**	**69-71**	**72-74**
No.of Persons	**:**	**20**	**113**	**138**	**130**	**19**

Solution:

Weight	No. of persons	c.f.
60-62	20	20
63-65	113	133
66-68	138	271
69-71	130	401
72-74	19	420
	N=Sf = 420	

Here $\frac{N}{2} = \frac{420}{2} = 210$

The cumulative frequency (c.f.) just greater than $\frac{N}{2} = 210$ is 271 and the corresponding Median class 66 - 68. However this should be changed to 65.5 - 68.5

$\Rightarrow l = 65.5$, m = 133, f = 138, c = 3

$$\therefore \text{Median} = l + \left(\frac{\frac{N}{2} - m}{f} \times c \right)$$

$$= 65.5 + \left(\frac{210 - 133}{138} \times 3 \right) = 67.2 \text{ kgs.}$$

10.1.3 Mode for continous frequency distribution

In case of continuous frequency distribution, mode is obtained by the following formula.

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - (f_0 + f_2)} \times c \right)$$

where l = lower limit of the modal class.

f_1 = frequency of the modal class.

f_0 = frequency of the class just preceding the modal class.

f_2 = frequency of the class just succeeding the modal class.

c = class magnitude or the length of the class interval corresponding to the modal class.

Observation :

Some times mode is estimated from the mean and the median. For a symmetrical distribution, mean, median and mode coincide. If the distribution is moderately asymmetrical the mean, median and mode obey the following empirical relationship due to Karl Pearson.

Mean - mode = 3(mean - median)

=> mode = 3 median - 2mean.

Example 7

Calculate the mode for the following data

Daily wages (in Rs.) :	**50-60**	**60-70**	**70-80**	**80-90**	**90-100**
No. of Workers :	**35**	**60**	**78**	**110**	**80**

Solution :

The greatest frequency = 110, which occurs in the class interval 80-90, so modal class interval is 80-90.

$\therefore l = 80, f_1 = 110, f_0 = 78; f_2 = 80; c = 10.$

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - (f_0 + f_2)} \times c\right)$$

$$= 80 + \left(\frac{110 - 78}{2(110) - (78 + 80)} \times 10\right)$$

$$= \text{Rs. } 85.16$$

10.1.4 Geometric Mean

(i) Geometric mean of n values is the n th root of the product of the n values. That is for the set of n individual observations $x_1, x_2 \ldots x_n$ their Geometric mean, denoted by G is

$$\sqrt[n]{x_1 \cdot x_2 \cdot x_3 \ldots x_n} \quad \text{or} \quad (x_1.\, x_2 \ldots x_n)^{1/n}$$

Observation:

$$\log G = \log (x_1, x_2 \ldots\ldots\ldots x_n)^{1/n}$$

$$= \frac{1}{n} \log (x_1, x_2 \ldots\ldots\ldots x_n)$$

$$\log G = \frac{1}{n} \sum_{i=1}^{n} \log x_i$$

$$\Rightarrow \log G = \frac{\Sigma \log x}{n}$$

$$\therefore \text{Geometric Mean} = G = \text{Antilog} \left(\frac{\Sigma \log x}{n}\right)$$

Example : 8

Find the Geometric Mean of 3, 6, 24, 48.

Solution :

Let x denotes the given observation.

x	log x
3	0.4771
6	0.7782
24	1.3802
48	1.6812
	Σ log x= 4.3167

G.M. = 11.99

(ii) In case of discrete frequency distrisbution i.e. if $x_1, x_2....x_n$ occur f_1, f_2, f_n times respectively, the Geometric Mean, G is given by

$$G = \left(x_1^{f_1}\ x_2^{f_2}\ \ldots\ x_n^{f_n}\right)^{\frac{1}{N}}$$

where $N = \Sigma f = f_1 + f_2 + \ldots + f_n$

Observation:

$$\log G = \frac{1}{N}\log\left(x_1^{f_1}\ x_2^{f_2}\ \ldots\ x_n^{f_n}\right)$$

$$= \frac{1}{N}[f_1 \log x_1 + f_2 \log x_2 + ... + f_n \log x_n]$$

$$= \frac{1}{N}\Sigma f_i \log x_i$$

$$\Rightarrow \log G = \frac{Óf_i \log x_i}{N}$$

$$\therefore G = \text{Antilog}\left(\frac{Óf_i \log x_i}{N}\right)$$

Example 9

Calculate Geometric mean for the data given below

x	**:**	**10**	**15**	**25**	**40**	**50**
f	**:**	**4**	**6**	**10**	**7**	**3**

Solution :

x	f	**log x**	**f logx**
10	4	1.0000	4.0000
15	6	1.1761	7.0566
25	10	1.3979	13.9790
40	7	1.6021	11.2147
50	3	1.6990	5.0970
		N = Sf = 30	**Sf logx = 41.3473**

$\therefore$ G $= \text{Antilog}\left(\frac{\text{Óf logx}}{\text{N}}\right)$

$= \text{Antilog}\left(\frac{41.3473}{30}\right)$

= Antilog (1.3782)

= 23.89

(iii) In the case of continuous frequency distribution,

$\therefore$ G $= \text{Antilog}\left(\frac{\text{Óf logx}}{\text{N}}\right)$

where $N = \Sigma f$ and x being the midvalues of the class intervals

Example 10

Compute the Geometric mean of the following data

Marks	**:**	**0-10**	**10-20**	**20-30**	**30-40**	**40-50**
No. of students	**:**	**5**	**7**	**15**	**25**	**8**

Solution :

Marks	**No. of Students f**	**Mid value x**	**log x**	**f log x**
0 – 10	5	5	0.6990	3.4950
10 – 20	7	15	1.1761	8.2327
20 – 30	15	25	1.3979	20.9685
30 – 40	25	35	1.5441	38.6025
40 – 50	8	45	1. 6532	13.2256
	N = Σf = 60			**Σf log x = 84.5243**

$\therefore$ G $= \text{Antilog}\left(\frac{\Sigma f \log x}{N}\right)$

$= \text{Antilog}\left(\frac{84.5243}{60}\right)$

$=$ Antilog (1.4087) = 25.63

Observation:

Geometric Mean is always smaller than arithmetic mean i.e. G.M. ≤ A.M. for a given data

10.1.5 Harmonic Mean

(i) Harmonic mean of a number of observations is the reciprocal of the arithmetic mean of their reciprocals. It is denoted by H.

Thus, if $x_1, x_2 \ldots x_n$ are the observations, their reciprocals are $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$. The total of the reciprocals is $= \Sigma\left(\frac{1}{x}\right)$ and the mean of the reciprocals is $= \frac{\Sigma \frac{1}{x}}{n}$

$\therefore$ the reciprocal of the mean of the reciprocals is $= \frac{n}{\Sigma\left(\frac{1}{x}\right)}$

$$H = \frac{n}{\Sigma\left(\frac{1}{x}\right)}$$

Find the Harmonic Mean of 6, 14, 21, 30

Solution :

x	$\frac{1}{x}$
6	0.1667
14	0.0714
21	0.0476
30	0.0333
	S $\frac{1}{x}$ = 0.3190

$$H = \frac{n}{\Sigma \frac{1}{x}} = \frac{4}{0.3190} = 12.54$$

$\therefore$ Harmonic mean is $H = 12.54$

(ii) In case of discrete frequency distribution, i.e. if x_1, x_2.....x_n occur f_1, f_2, ...f_n times respectively, the Harmonic mean, H is given by

$$H = \frac{1}{\frac{\frac{f_1}{x_1}+\frac{f_2}{x_2}+\ldots+\frac{f_n}{x_n}}{N}} = \frac{1}{\frac{1}{N}\Sigma\left(\frac{f}{x}\right)} = \frac{N}{\Sigma\left(\frac{f}{x}\right)}$$

where $N = \Sigma f$

Example 12

Calculate the Harmonic mean from the following data

x :	**10**	**12**	**14**	**16**	**18**	**20**
f :	**5**	**18**	**20**	**10**	**6**	**1**

Solution :

x	**f**	$\mathbf{\frac{f}{x}}$
10	5	0.5000
12	18	1.5000
14	20	1.4286
16	10	0.6250
18	6	0.3333
20	1	0.0500
	N = Sf= 60	**S $\mathbf{\frac{f}{x}}$ = 4.4369**

$$H = \frac{N}{\Sigma\left(\frac{f}{X}\right)}$$

$$= \frac{60}{4.4369} = 13.52$$

(iii) The Harmonic Mean for continuous frequency distribution is given by H = $\frac{N}{Ó\left(\frac{f}{x}\right)}$, where N = Sf and x = mid values of the class intervals

Example 13

Calculate the Harmonic mean for the following data.

Size of items	**50-60**	**60-70**	**70-80**	**80-90**	**90-100**
No. of items	**12**	**15**	**22**	**18**	**10**

Solution :

size	**f**	**x**	$\frac{f}{x}$
50-60	12	35	0.2182
60-70	15	65	0.2308
70-80	22	75	0.2933
80-90	18	85	0.2118
90-100	10	95	0.1053
	N = S f = 77		**S $\frac{f}{x}$ = 1.0594**

$$H = \frac{N}{\Sigma \frac{f}{x}} = \frac{77}{1.0594} = 72.683$$

Observation:

(i) For a given data H.M. ≤ G.M.

(ii) H.M. ≤ G.M. ≤ A.M.

(iii) (A.M.) x (H.M.) = $(G.M.)^2$

EXERCISE 10.1

1) Find the arithmetic mean of the following set of observation
25, 32, 28, 34, 24, 31, 36, 27, 29, 30.

2) Calculate the arithmetic mean for the following data.

Age in Years	:	8	10	12	15	18
No.of Workers	:	5	7	12	6	10

3) Calculate the arithmetic mean of number of persons per house. Given

No. of persons per house	:	2	3	4	5	6
No. of houses	:	10	25	30	25	10

4) Calculate the arithmetic mean by using deviation method.

Marks	: 40	50	54	60	68	80
No. of Students	: 10	18	20	39	15	8

5) From the following data, compute arithmetic mean, median and evaluate the mode using emprical relation

Marks :	0-10	10-20	20-30	30-40	40-50	50-60
No. of Students :	5	10	25	30	20	10

6) Find the arithmetic mean, median and mode for the following frequency distribution.

Class limits:	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89
Frequency :	5	9	14	20	25	15	8	4

7) Find the median of the following set of observations.
37, 32, 45, 36, 39, 31, 46, 57, 27, 34, 28, 30, 21

8) Find the median of 57, 58, 61, 42, 38, 65, 72, 66.

9) Find the median of the following frequency distribution.

Daily wages (Rs.) :	5	10	15	20	25	30
No. of Persons (f) :	7	12	37	25	22	11

10) The marks obtained by 60 students are given below. Find the median.

Marks (out of 10):	3	4	5	6	7	8	9	10
No. of Students:	1	5	6	7	10	15	11	5

11) Calculate the median from the following data.

Marks :	10-25	25-40	40-55	55-70	70-85	85-100
Frequency :	6	20	44	26	3	1

12) Find the median for the following data.

Class limits :	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency :	3	7	13	17	12	10	8	8	6	6

13) Find the mode for the following set of observations.
41, 50, 75, 91, 95, 69, 61, 53, 69, 70, 82, 46, 69.

14) Find the mode of the following:

Size of Dress :	22	28	30	32	34
No. of sets produced:	10	22	48	102	55

15) Calculate the mode from the following

Size :	10	11	12	13	14	15	16	17	18
Frequency :	10	12	15	19	20	8	4	3	2

16) Find the mode of the following distribution.

Class limits:	10-15	15-20	20-25	25-30	30-35	35-40	40-45	45-50
Frequency :	4	12	16	22	10	8	6	4

17) Calculate the Geometric Mean for the following data.
35, 386, 153, 125, 118, 1246

18) Calculate the Geometric Mean for the following data.

Value :	10	12	15	20	50
Frequency :	2	3	10	8	2

19) The following distribution relates to marks in Accountancy of 60 students.

Marks :	0-10	10-20	20-30	30-40	40-50	50-60
Students :	3	8	15	20	10	4

Find the Geometric Mean

20) Calculate the Harmonic mean for the following data.
2, 4, 6, 8 10

21) Calculate the Harmonic mean.

Size :	6	7	8	9	10	11
Frequency :	4	6	9	5	2	8

22) From the following data, compute the value of Harmonic mean.

Class interval:	10-20	20-30	30-40	40-50	50-60
Frequency :	4	6	10	7	3

10.2 MEASURES OF DISPERSION

"Dispersion is the measure of variation of the items" - A.L.Bowley

In a group of individual items, all the items are not equal. There is difference or variation among the items. For example, if we observe the marks obtained by a group of studens, it could be easily found the difference or variation among the marks.

The common averages or measures of central tendency which we discussed earlier indicate the general magnitude of the data but they do not reveal the degree of variability in individual items in a group or a distribution. So to evaluate the degree of variation among the data, certain other measures called, measures of dispersion is used.

Measures of Dispersion in particular helps in finding out the variability or Dispersion/Scatteredness of individual items in a given distribution. The variability (Dispersion or Scatteredness) of the data may be known with reference to the central value (Common Average) or any arbitrary value or with reference to other vaues in the distribution. The mean or even Median and Mode may be same in two or more distribution, but the composition of individual items in the series may vary widely. For example, consider the following marks of two students.

Student I	**Student II**
68	82
72	90
63	82
67	21
70	65
340	340
Average 68	Average 68

It would be wrong to conclude that performance of two students is the same, because of the fact that the second student has failed in one paper. Also it may be noted that the variation among the marks of first student is less than the variation among the marks of the second student. Since less variation is a desirable characteristic, the first student is almost equally good in all the subjects.

It is thus clear that **measures of central tendency are insufficient to reveal the true nature and important characteristics of the data.** Therefore we need some other measures, called measures of Dispersion. Few of them are Range, Standard Deviation and coefficient of variation.

10.2.1 Range

Range is the difference between the largest and the smallest of the values.

Symbollically,

Range = L - S

where L = Largest value

S = Smallest value

Co-efficient of Range is given by = $\frac{L - S}{L + S}$

Example 14

Find the value of range and its coefficient for the following data

6 8 5 10 11 12

Solution:

L = 12 (Largest)

S = 5 (Smallest)

∴ Range = L - S = 7

Co-efficient of Range = $\frac{L - S}{L + S}$ = 0.4118

Example : 15

Calculate range and its coefficient from the following distribution.

Size	**20 - 22**	**23 - 25**	**26 - 28**	**29 - 31**	**32 - 34**
Number	**7**	**9**	**19**	**42**	**27**

Solution:

Given is a continuous distribution. Hence the following method is adopted.

Here, L = Midvalue of the highest class

∴ L = $\frac{32+34}{2}$ = 33

S = Mid value of the lowest class

∴ S = $\frac{20+22}{2}$ = 21

∴ Range = L - S = 12

Co-efficient of Range = $\frac{L-S}{L+S}$ = 0.22

10.2.2 Standard Deviation

Standard Deviation is the root mean square deviation of the values from their arithmetic mean.

S.D. is the abbreviation of standard Deviation and it is represented by the symbol σ (read as sigma). The square of standard deviation is called variance denoted by σ^2

(i) Standard Deviation for the raw data.

$$\sigma = \sqrt{\frac{\Sigma d^2}{n}}$$

Where d $= x - \bar{x}$

n = number of observations.

Example 16

Find the standard deviation for the following data

75, 73, 70, 77, 72, 75, 76, 72, 74, 76

Solution :

x	$d = x - \bar{x}$	d^2
75	1	1
73	-1	1
70	-4	16
77	3	9
72	-2	4
75	1	1
76	2	4
72	-2	4
74	0	0
76	2	4
Sx = 740	**Sd = 0**	**Sd² = 44**

$$\bar{x} = \frac{\Sigma x}{n} = \frac{740}{10} = 74$$

∴ Standard Deviation,

$$\sigma = \sqrt{\frac{\Sigma d^2}{n}} = \sqrt{\frac{44}{10}} = 2.09$$

(ii) Standard deviation for the raw data without using Arithmetic mean.

The formula to calculate S.D in this case

$$\sigma = \sqrt{\left(\frac{\Sigma x^2}{n}\right) - \left(\frac{\Sigma x}{n}\right)^2}$$

Example : 17

Find the standard deviation of the following set of observations.
1, 3, 5, 4, 6, 7, 9, 10, 2.

Solution :

Let x denotes the given observations

x	:	1	3	5	4	6	7	9	8	10	2
x^2	:	1	9	25	16	36	49	81	64	100	4

Here $\Sigma x = 55$

$\Sigma x^2 = 385$

$$\therefore \sigma = \sqrt{\left(\frac{\Sigma x^2}{n}\right) - \left(\frac{\Sigma x}{n}\right)^2}$$

$$= \sqrt{\left(\frac{385}{10}\right) - \left(\frac{55}{10}\right)^2} = 2.87$$

(iii) S.D. for the raw data by Deviation Method

By assuming arbitrary constant, A, the standard deviation is given by

$$\sigma = \sqrt{\left(\frac{\Sigma d^2}{n}\right) - \left(\frac{\Sigma d}{n}\right)^2}$$

where d = x - A

A = arbitrary constant

Σd^2 = Sum of the squares of deviations

Σd = sum of the deviations

n = number of observations

Example 18

For the data given below, calculate standard deviation
25, 32, 53, 62, 41, 59, 48, 31, 33, 24.

Solution:

Taking A = 41

x:	25	32	53	62	41	59	48	31	33	24
d = **x -A**	-16	-9	12	21	0	18	7	-10	-8	-17
d^2	256	81	144	441	0	324	49	100	64	289

Here $\Sigma d = -2$

$\Sigma d^2 = 1748$

$$\sigma = \sqrt{\left(\frac{\Sigma d^2}{n}\right) - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\left(\frac{1748}{10}\right) - \left(\frac{-2}{10}\right)^2} = 13.21$$

(iv) Standard deviation for the discrete grouped data

In this case

$$\sigma = \sqrt{\frac{\Sigma fd^2}{N}} \quad \text{where } d = x - \overline{X}$$

Example 19

Calculate the standard deviation for the following data

x	**6**	**9**	**12**	**15**	**18**
f:	**7**	**12**	**13**	**10**	**8**

Solution:

x	f	fx	d = x - $\overline{X}$	d^2	fd^2
6	7	42	-6	36	252
9	12	108	-3	9	108
12	13	156	0	0	0
15	10	150	3	9	90
18	8	144	6	36	288
N=Sf =50		**Sfx = 600**			**Sfd² = 738**

$$\overline{X} = \frac{\Sigma fx}{N} = \frac{600}{50} = 12$$

$$\sigma = \sqrt{\frac{\Sigma fd^2}{N}} = \sqrt{\frac{738}{50}} = 3.84$$

(v) Standard deviation for the continuous grouped data without using Assumed Mean.

In this case

$$\sigma = c \times \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} \quad \text{where } d = \frac{x - A}{c}$$

Example 20

Compute the standard deviation for the following data

Class interval :	**0-10**	**10-20**	**20-30**	**30-40**	**40-50**	**50-60**	**60-70**
Frequency :	**8**	**12**	**17**	**14**	**9**	**7**	**4**

Solution :

Taking A = 35

Class Intervals	Frequency f	Mid value x	$d = \frac{x-A}{c}$	fd	fd^2
0-10	8	5	-3	-24	72
10-20	12	15	-2	-24	48
20-30	17	25	-1	-17	17
30-40	14	A35	0	0	0
40-50	9	45	1	9	9
50-60	7	55	2	14	28
60-70	4	65	3	12	36
	N= Sf = 71			**Sfd=-30**	**Sfd²=210**

$$\sigma = c \text{ x} \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2}$$

$$= 10 \text{ x} \sqrt{\frac{210}{71} - \left(\frac{-30}{71}\right)^2}$$

$$= 16.67$$

10.2.3. Coefficient of variation

Co-efficient of variation denoted by C.V. and is given by

$$C.V. = \left(\frac{\sigma}{x} \text{ x } 100\right)\%$$

Observation:

(i) Co-efficient of variation is a **percentage expression**, it is used to compare two or more groups.

(ii) The group which has less coefficient of variation is said to be **more consistent** or **more stable,** and the group which has more co-efficient of variation is said to be **more variable** or **less consistent.**

Example 21

Prices of a particular commodity in two cities are given below.

City A :	**40**	**80**	**70**	**48**	**52**	**72**	**68**	**56**	**64**	**60**
City B :	**52**	**75**	**55**	**60**	**63**	**69**	**72**	**51**	**57**	**66**

Which city has more stable price

Solution :

City A	City B	$d_x = x - \bar{x}$	$d_y = y - \bar{y}$	$d_x^2 = (x-\bar{x})^2$	$d_y^2 = (y-\bar{y})^2$
40	52	-21	-10	441	100
80	75	19	13	361	169
70	55	9	-7	81	49
48	60	-13	-2	169	4
52	63	-9	1	81	1
72	69	11	7	121	49
68	72	7	10	49	100
56	51	-5	-11	25	121
64	57	3	-5	9	25
60	66	-1	-4	1	16
Sx=610	**Sy=620**			**Sd²ₓ = 1338**	**Sd²ᵧ = 634**

$$\bar{x} = \frac{\Sigma x}{n} = \frac{610}{10} = 61$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{620}{10} = 62$$

$$\sigma_x = \sqrt{\frac{1338}{10}} = 11.57$$

$$\sigma_y = \sqrt{\frac{634}{10}} = 7.96$$

$$\text{C.V. (x)} = \frac{\sigma_x}{\bar{x}} \times 100$$

$$= \frac{11.57}{61} = 18.97\%$$

$$\text{C.V. (y)} = \frac{\sigma_y}{\bar{y}} \times 100$$

$$= \frac{7.96}{62}\ 12.84\%$$

Conclusion

Comparatively, C.V. (y) < C.V (x)

$\Rightarrow$ City B has more stable price.

EXERCISES 10.2

1) Find the range and co-efficient of range for the following data.
   a) 12, 8, 9, 10, 4, 14, 15
   b) 35, 40, 52, 29, 51, 46, 27, 30, 30, 23.

2) Calculate range and its Co-efficient from the following distribution.

Size :	60-62	63-65	66-68	69-71	72-74
Number :	5	18	42	27	8

3) Find the range and its co-efficient from the following data.

Wages (in Rs) :	35-45	45-55	55-65	65-75	75-85
No.of Workers :	18	22	30	6	4

4) Find the standard deviation of the set of numbers
   3, 8, 6, 10, 12, 9, 11, 10, 12, 7.

5) Find the S.D. of the following set of observations by using Deviation Method.
   45, 36, 40, 36, 39, 42, 45, 35, 40, 39.

6) Find the S.D. from the following set of observation by using i) Mean ii) Deviation method iii) Direct Method.
   25, 32, 43, 53, 62, 59, 48, 31, 24, 33

7) Find the standard Deviation for the following data

x :	1	2	3	4	5
f :	3	7	10	3	2

8) Calculate the standard deviation for the following

No.of Goals Scored in a Match :	0	1	2	3	4	5
No.of Matches :	1	2	4	3	0	2

9) Calculate the S.D. for the following continous frequency distribution.

Class interval:	4-6	6-8	8-10	10-12	12-14
Frequency :	10	17	32	21	20

10) Calculate the S.D. of the following frequency distribution.

Annual profit (Rs.Crores):	20-40	40-60	60-80	80-100
No.of Banks :	10	14	25	48

Annual profit (Rs.Crores):	100-120	120-140	140-160
No.of Banks	33	24	16

11) Calculate the co-efficient of variation of the following
40 41 45 49 50 51 55 59 60 60

12) From the following price of gold in a week, find the city in which the price was more stable.

City A :	498	500	505	504	502	509
City B :	500	505	502	498	496	505

13) From the following data, find out which share is more stable in its value.

x :	55	54	52	53	56	58	52	50	51	49
y:	108	107	105	105	106	107	104	103	104	101

10.3. CONCEPT OF PROBABILITY

Consider the following experiment

(i) A ball is dropped from a certain height.

(ii) A spoon full of sugar is added to a cup of milk.

(iii) Petrol is poured over fire.

In each of the above experiments, the result or outcome is **certain**, and is known in advance. That is, in experiment (i), the ball is certain to touch the earth and in (ii) the sugar will certainly dissolve in milk and in (iii) the petrol is sure to burn.

But in some of the experiments such as

(i) spinning a roulette wheel

(ii) drawing a card from a pack of cards.

(iii) tossing a coin

(iv) throwing a die etc.,

in which the result is **uncertain**.

For example, when a coin is tossed everyone knows that there are two possible out comes, namely head or tail. But no one could say with certainty which of the two possible outcomes will be obtained. Similarly, in throwing a die we know that there are six possible outcomes 1 or 2 or 3 or ... 6. But we are not sure of what out come will really be.

In all, such experiments, that there is an **element of chance**, called **probability** which express the element of chance numerically.

The theory of probability was introduced to give a **quantification** to the possibility of certain outcome of the experiment in the face of **uncertainty.**

Probability, one of the fundamental tools of statistics, had its formal beginning with **games of chance** in the seventeenth century. But soon its application in all fields of study became obvious and it has been extensively used in all fields of human activity.

10.3.1 Basic Concepts

(i) Random Experiment

Any operation with outcomes is called an **experiment.**

A Random experiment is an experiment.

(i) in which all outcomes of the experiment are known in advance.

(ii) what specific (particular) outcome will result is not known in advance, and

(iii) the experiment can be repeated under identical (same) conditions.

(ii) Event

All possible outcomes of an experiment are known as events.

(iii) Sample Space

The set of all possible outcomes of an experiment is known as sample space of that experiment and is denoted by S.

(iv) Mutually Exclusive events

Events are said to be mutually exclusive if the occurrence of one prevents the occurrence of all other events. That is two or more mutually exclusive events cannot occur simultaneously, in the same experiment.

For example

Consider the following events A and B in the experiment of drawing a card from the pack of 52 cards.

A : The card is spade

B : The card is hearts.

These two events A and B are mutually exclusive. Since a card drawn cannot be both a spade and a hearts.

(v) Independent events

Events (two or more) are said to be independent if the occurrence or non-occurrence of one does not affect the occurrence of the others.

For example

Consider the experiment of tossing a fair coin. The occurrence of the event Head in the first toss is independent of the occurrence of the event Head in the second toss, third toss and subsequent tosses.

(vi) Complementary Event

The event 'A occurs' and the event 'A does not occur' are called complementary events. The event 'A does not occur' is denoted by A^C or $\overline{A}$ or A' and read as complement of A.

(vii) Equally likely

Events (two or more) of an experiment are said to be equally likely, if any one them cannot be expected to occur in preference to the others.

(viii) Favourable events or cases

The number of outcomes or cases which entail the occurrence of the event in an experiment is called favourable events or favourable cases.

For example

consider the experiment in which Two fair dice are rolled.

In this experiment, the number of cases favorable to the event of getting a sum 7 is : (1,6) (6,1) (5,2) (2,5), (3,4), (4,3).

That is there are 6 cases favorable to an event of sum = 7.

(ix) Exhaustive Events

The totality of all possible outcomes of any experiment is called an exhaustive events or exhaustive cases.

10.3.2 Classical Definition of Probability

If an experiment results in **n** exhaustive, mutually exclusive and equally likely cases and **m** of them are favourable to the occurence of an event A, then the ratio **m/n** is called the probability of occurence of the event A, denoted by P(A).

$$\therefore \mathbf{P\,(A) = \frac{m}{n}}$$

Observation :

(i) $O \leq P(A) \leq 1$

(ii) If P(A) = 0, then A is an impossible event.

The number of favourable cases (m) to the event A, cannot be greater than the total number of exhaustive cases (n).

That is $0 \leq m \leq n$

$\Rightarrow 0 \leq \frac{m}{n} \leq 1$

(iii) For the sample space S, P(S) = 1. S is called sure event.

Example 22

A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue?

Solution:

Total number of balls = 3 + 6 + 7 = 16

Then out of 16 balls, 2 balls can be drawn in $^{16}C_2$ ways.

$\therefore$ $n = {}^{16}C_2 = 120$

Let A be the event that the two balls drawn are white and blue.

Since there are 6 white balls and 7 blue balls, the total number of cases favourable to the event A is $^{6}C_1 \times {}^{7}C_1 = 6 \times 7 = 42$

i.e. m = 42

$\therefore$ $P(A) = \frac{m}{n} = \frac{42}{120} = \frac{7}{20}$

Example 23

A coin is tossed twice. Find the probability of getting atleast one head.

Solution:

Here the sample space is S = {(H,H), (H,T), (T,H), (T,T)}

$\therefore$ The total no. of possible outcomes n = 4

The favourable outcomes for the event ‘at least one head’ are (H,H), (H,T), (T.H).

$\therefore$ No. of favourable outcomes m = 3

$\therefore$ P (getting atleast one head) = $\frac{3}{4}$

Example 24

An integer is chosen at random out of the integers 1 to 100. What is the probability that it is i) a multiple of 5 ii) divisible by 7 iii) greater than 70.

Solution:

Total number of possible outcomes $= {}^{100}C_1 = 100$

(i) The favourable outcomes for the event "Multiple of 5" are (5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55…..100)

$\therefore$ No. of favourable outcomes $= {}^{20}C_1 = 20$

$\therefore$ P (that chosen number is a multiple of 5) $= \frac{20}{100} = \frac{1}{5}$

(ii) The favourable outcomes for the event 'divisible by 7' are (7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98)

$\therefore$ No. of favourable outcomes $= {}^{14}C_1 = 14$

$\therefore$ P (that chosen number is divisible by 7) $= \frac{14}{100} = \frac{7}{50}$

(iii) No. of favourable outcomes to the event 'greater than 70' = 30

$\therefore$ P (that chosen number is greater than 70) $= \frac{30}{100} = \frac{3}{10}$

10.3.3 Modern Definition of Probability

The modern approach to probability is purely axiomatic and it is based on the set theory.

In order to study the theory of probability with an axiomatic approach, it is necessary to define certain basic concepts. They are

(i) Sample space: Each possible outcome of an experiment that can be repeated under similar or identical conditions is called a sample point and the totality of sample points is called the sample space, denoted by S.

(ii) Event:

Any subset of a sample space is called an event.

(iii) Mutually Exclusive Events:

Two events A and B are said to be mutually exclusive events if $A \cap B = \varphi$, i.e. if, A and B are disjoint sets.

For example,

consider S = {1,2,3,4,5}

Let A = the set of odd numbers = {1,3,5} and

B = the set of even numbers = {2,4}

Then $A \cap B = \varphi$

∴ events A and B are mutually exclusive.

Observation:

Statement Meaning interms of set theory

(i) $A \cup B$ => Atleast one of the events A or B occurs

(ii) $A \cap B$ => Both the events A and B occur

(iii) $\overline{A} \cap \overline{B}$ => Neither A nor B occurs

(iv) $A \cap \overline{B}$ => Event A occurs and B does not occur

10.3.4 Definition of Probability (Axiomatic)

Let E be an experiment. Let S be a sample space associated with E. With every event in S we associate a real number denoted by P(A), called the probability of the event A satisfying the following axioms.

Axiom1. $P(A) \geq 0$

Axiom2. $P(S) = 1$

Axiom3. If A_1, A_2 ... is a sequence of mutually exclusive events in S then

$P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$

Example 25

Let a sample space be S = {w_1, w_2, w_3}. Which of the following defines probability space on S?

(i) $P(w_1) = 1$, $P(w_2) = \frac{2}{3}$ $P(w_3) = \frac{1}{3}$

(ii) $P(w_1) = \frac{2}{3}$, $P(w_2) = \frac{1}{3}$, $P(w_3) = -\frac{2}{3}$

(iii) $P(w_1) = 0$, $P(w_2) = \frac{2}{3}$ $P(w_3) = \frac{1}{3}$

Solution:

(i) Here each $P(w_1)$, $P(w_2)$ and $P(w_3)$ are non-negative.

ie: $P(w_1) \geq 0$, $P(w_2) \geq 0$, $P(w_3) \geq 0$.

But $P(w_1) + P(w_2) + P(w_3) \neq 1$

So by axiom 2, this set of probability functions does not define a probability space on S.

(ii) Since $P(w_3)$ is negative by axiom 1 the set of probability function does not define a probability space on S.

(iii) Here all probabilities, $P(w_1)$, $P(w_2)$ and $P(w_3)$ are non-negative.

Also $P(w_1) + P(w_2) + P(w_3) = 0 + \frac{2}{3} + \frac{1}{3} = 1$

$\therefore$ by axiom 1,2, the set of probability function defines a probability space on S.

Example 26

Let P be a probability function on S = $\{w_1, w_2, w_3\}$.

Find $P(w_2)$ if $P(w_1) = \frac{1}{3}$ and $P(w_3) = \frac{1}{2}$

Solution:

Here $P(w_1) = \frac{1}{3}$ and $P(w_3) = \frac{1}{2}$ are both non-negative.

By axiom 2,

$P(w_1) + P(w_2) + P(w_3) = 1$

$$\therefore P(w_2) = 1 - P(w_1) - P(w_3)$$

$$= 1 - \frac{1}{3} - \frac{1}{2}$$

$$= \frac{1}{6} \text{ which is non-negative.}$$

$$\Rightarrow P(w_2) = \frac{1}{6}$$

10.3.5 Basic Theorems on Probability of Events

Theorem : 1

Let S be the sample space. Then $P(\varphi) = o$. ie. probability of an impossible event is zero.

Proof:

We know that $S \cup \varphi = S$

$\therefore P(S \cup \varphi) = P(S)$

ie. $P(S) + P(\varphi) = P(S)$ by axiom 3. $\therefore P(\phi) = 0$

Theorem : 2

Let S be the sample space and A be an event in S

Then $P(\overline{A}) = 1 - P(A)$

Proof :

We know that $A \cup \overline{A} = S$

$\therefore P(A \cup \overline{A}) = P(S)$

$P(A) + P(\overline{A}) = 1$ by axiom (2) and (3)

$\Rightarrow P(\overline{A}) = 1 - P(A)$

10.3.6 Addition Theorem

Statement: If A and B are any two events, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Observation:

(i) If the two events A and B are mutually exclusive, then $A \cap B = \varphi$

$\therefore P(A \cap B) = 0$

$\Rightarrow P(A \cup B) = P(A) + P(B)$

(ii) The addition Theorem may be extended to any three events A,B,C and we have

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

Example: 27

A card is drawn from a well shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

Solution:

Total number of cards in a pack = 52.

$\therefore$ The sample space contains 52 sample points, and each and every sample points has the same probability (equal probability).

Let A be the event that the card drawn is a spade.

$\therefore P(A) = P(\text{that the drawn card is spade})$

$$= \frac{^{13}C_1}{^{52}C_1} \text{ since A consists of 13 sample ie: 13 spade cards.}$$

$$P(A) = \frac{13}{52}$$

Let B be the event that the card drawn is an ace.

$\therefore P(B) = P\ (\text{that the drawn card is an ace})$

$$= \frac{^{4}C_1}{^{52}C_1} \text{ since B consists of 4 sample points ie: 4 ace cards.}$$

$$= \frac{4}{52}$$

The compound event $(A \cap B)$ consists of only one sample point, the ace of spade.

$\therefore P(A \cap B) = P$ (that the card drawn is ace of spade)

$$= \frac{1}{52}$$

Hence, $P(A \cup B) = P$ (that the card drawn is either a spade or an ace)

$= P(A) + P(B) - P(A \cap B)$ (by addition theorem)

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

$$\Rightarrow P(A \cup B) = \frac{4}{13}$$

Example 28

One number, out of 1 to 20 number, is selected at random. What is the probability that it is either a multiple of 3 or 4

Solution:

One number is selected at random and that can be done in $^{20}C_1$ ways.

ie: Sample space S consists of 20 sample points.

$\Rightarrow S = \{1,2,3,... 20\}$

Let A be the event that the number chosen is multiple of 3.

Then A = {3,6,9,12,15,18}

$\therefore$ P (A) = P (that the selected number is multiple of 3} = $\frac{6}{20}$

Let B be the event that the number choose is Multiple of 4.

Then B = {4,8,12,16,20}

P(B) = P (that the selected number is multiple of 4) = $\frac{5}{20}$

The event $A \cap B$ consists of only one sample point 12, which is a multiple of 3 and multiple of 4.

$\Rightarrow A \cap B = \{12\}$

$P(A \cap B)$ = P (that the selected number is multiple of 3 and multiple of 4}

$= \frac{1}{20}$

Hence

$P(A \cup B)$ = P (that the selected number is either multiple of 3 or multiple of 4)

$= P(A) + P(B) - P(A \cap B)$

$= \frac{6}{20} + \frac{5}{20} - \frac{1}{20} = \frac{10}{20}$

$P(A \cup B) = \frac{1}{2}$

Example 29

A bag contains 6 black and 5 red balls. Two balls are drawn at random. What is the probability that they are of the same colour.

Solution:

Total numberof balls = 11

number of balls drawn = 2

$\therefore$ Exhaustive number of cases = ${}^{11}C_2 = 55$

Let A be the event of getting both the balls are black and B be the event of getting both the balls are red.

Hence by addition theorem of probability, required probability.

P (two balls are of same colour) = P(AUB)

= P(A) + P(B)

$$= \frac{{}^{6}c_2}{{}^{11}c_2} + \frac{{}^{5}c_2}{{}^{11}c_2}$$

$$= \frac{15}{55} + \frac{10}{55} = \frac{25}{55} = \frac{5}{11}$$

Example 30

A box contains 6 Red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is atleast one ball of each colour.

Solution :

Total no. of balls = 15

Number of balls drawn = 4

$\therefore$ Exhaustive number of cases = ${}^{15}c_4$ = 1365

The required event E that there is atleast one ball of each colour among the 4 drawn from the box at random can occur in the following mutually disjoint ways. (R, W, B denotes Red, White and Black balls)

E = (R = 1, W = 1, B = 2) U (R = 2, W = 1, B = 1) U (R = 1, W = 2, B = 1)
Hence by addition theorem of probability,

P(E) = P(R=1, W=1, B=2) + P(R=2, W=1, B=1) + P(R=1, W=2, B=1)

$$= \frac{{}^{6}c_1 \text{ x } {}^{4}c_1 \text{ x } {}^{5}c_2}{{}^{15}c_4} + \frac{{}^{6}c_2 \text{ x } {}^{4}c_1 \text{ x } {}^{5}c_1}{{}^{15}c_4} + \frac{{}^{6}c_1 \text{ x } {}^{4}c_2 \text{ x } {}^{5}c_1}{{}^{15}c_4}$$

$$= \frac{1}{{}^{15}c_4} [(6 \text{ x } 4 \text{ x } 10) + (15 \text{ x } 4 \text{ x } 5) + (6 \text{ x } 6 \text{ x } 5)]$$

$$= \frac{1}{{}^{15}c_4} [240 + 300 + 180] = \frac{720}{1365} = \frac{48}{91}$$

10.3.7 Conditional Probability

Definition:

Let A and B be two events in a sample space S. The conditional probability of the event B given that A has occurred is defined by

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0.$$

Observation:

(i) Similarly $P(A/B) = \frac{P(A \cap B)}{P(B)}$, if $P(B) \neq 0$.

(ii) Whenever we compute P(A/B), P(B/A) we are essentially computing it with respect to the restricted sample space.

Example: 31

Three fair coins are tossed. If the first coin shows a tail, find the probability of getting all tails

Solution:

The experiment of tossing three fair coins results the sample space.

S = {(HHH), (HHT), (HTH), (THH), (THT), (HTT), (TTH), (TTT)}

$\Rightarrow$ n(S) = 8.

Event A = the first coin shows a tail

= {(THH), (THT), (TTH), (TTT)}

n(A) = 4.

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

Let B be the event denotes getting all tails: ie:(TTT).

Let $B \cap A$ denotes the compound event of getting all tails and that the first coin shows tail.

$\Rightarrow \therefore$ $B \cap A$ = {(TTT)}

$n(B \cap A)$ = 1

$$\therefore \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{8} \text{ since } B \cap A = A \cap B.$$

Hence by formula.

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\therefore P(B/A) = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{2}{8} = \frac{1}{4}$$

Example: 32

A box contains 4 red and 6 green balls. Two balls are picked out one by one at random without replacement. What is the probability that the second ball is green given that the first one is green

Solution:

Define the following events.

A = {the first ball drawn is green}

B = {the second ball drawn is green}

Total number of balls = 4+6 = 10

Two balls are picked out at random one by one.

Here we have to compute P(B/A).

When the first ball is drawn,

P(A) = P(that the first ball drawn is green)

$$= \frac{{}^{6}C_1}{{}^{10}C_1} = \frac{6}{10}$$

Since the first ball(green) pickedout is not replaced, total number of balls in a box gets reduced to 9 and the total number of green balls reduced to 5.

$$\therefore P(A \cap B) = \frac{{}^{6}C_1}{{}^{10}C_1} \times \frac{{}^{5}C_1}{{}^{9}C_1} = \frac{6}{10} \times \frac{5}{9} = \frac{1}{3}$$

Hence P(B/A) = P (that the second ball drawn is green given that the first ball drawn is green)

$$= \frac{P(A \cap B)}{P(A)}$$

$$P(B/A) = \frac{\frac{1}{3}}{\frac{6}{10}} = \frac{1}{3} \times \frac{10}{6} = \frac{5}{9}$$

10.3.8 Multiplication Theorem for independent events

If A and B are two independent events then $P(A \cap B) = P(A)\,P(B)$

Observation:

For n independent events

$P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = P(A_1)\,P(A_2)\,P(A_3) \ldots P(A_n)$

Example 33

In a shotting test the probabilities of hitting the target are $\frac{1}{2}$ for A, $\frac{2}{3}$ for B and $\frac{3}{4}$ for C. If all of them fire at the same target, calculate the probabilities that

(i) all the three hit the target

(ii) only one of them hits the target

(iii) atleast one of them hits the target

Solution:

Here $P(A) = \frac{1}{2}$, $P(B) = \frac{2}{3}$, $P(C) = \frac{3}{4}$

$P(\overline{A}) = 1-\frac{1}{2} = \frac{1}{2}$, $P(\overline{B}) = 1-\frac{2}{3} = \frac{1}{3}$, $P(\overline{C}) = 1-\frac{3}{4} = \frac{1}{4}$

(i) P (all the three hit the target) $= P(A\cap B\cap C)$

$= P(A)\, P(B)\, P(C)$

($\because$ A, B, C hits independently)

$= \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4} = \frac{1}{4}$

Let us define the events

$E_1 =$ {only one of them hits the target}

$= \{(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)\}$

$E_2 =$ {atleast one of them hits the target}

$= \{(A \cup B \cup C)\}$

Hence

(ii) $P(E_1) = P(A \cap \overline{B} \cap \overline{C}) + P(\overline{A} \cap B \cap \overline{C}) + P(\overline{A} \cap \overline{B} \cap C)$

$= \frac{1}{2}\ \frac{1}{3}\ \frac{1}{4} + \frac{1}{2}\ \frac{2}{3}\ \frac{1}{4} + \frac{1}{2}\ \frac{1}{3}\ \frac{3}{4}$

$= \frac{1}{4}$

(iii) $P(E_2) = P(A\cup B\cup C)$

$= P(A) + P(B) + P(C) - P(A\cap C) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

$= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} - \frac{1}{2}\ \frac{2}{3} - \frac{2}{3}\ \frac{3}{4} - \frac{1}{2}\ \frac{3}{4} + \frac{1}{2}\ \frac{2}{3}\ \frac{3}{4}$

$= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} - \frac{1}{3} - \frac{1}{2} - \frac{3}{8} + \frac{1}{4}$

$= \frac{23}{24}$

Example 34

A problem is given to three students A, B, C whose chances of solving it are respectively $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. What is the probability that the problem is solved.

Solution:

P(A) = P(that A can solve the problem) = $\frac{1}{2}$

P(B) = P(that B can solve the problem) = $\frac{1}{3}$

P(C) = P(that C can solve the problem) = $\frac{1}{4}$

Since A, B, C are independent

$P(A\cap B) = P(A)\,P(B) = \frac{1}{2}\ \frac{1}{3}$

$P(B\cap C) = P(B)\,P(C) = \frac{1}{3}\ \frac{1}{4}$

$P(C\cap A) = P(C)\,P(A) = \frac{1}{4}\ \frac{1}{2}$

$P(A\cap B\cap C)= P(A)\,P(B)\,P(C) = \frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}$

$\therefore$ P(that the problem is solved) = P(that atleast one of them solves the problem)

= P (AUBUC)

$= P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2}\ \frac{1}{3} - \frac{1}{3}\ \frac{1}{4} - \frac{1}{2}\ \frac{1}{4} + \frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}$

$= \frac{12+8+6-4-2-3+1}{24} = \frac{18}{24} = \frac{3}{4}$

10.3.9 Baye's Theorem

Let S be a sample space

Let A_1, A_2, ... An be disjoint events in S and B be any arbitrary event in S with

$P(B) \neq 0$. Then Baye's theorem says

$$P(A_r/B) = \frac{P(A_r)\,P(B/A_r)}{\sum_{r=1}^{n} P(A_r)\,P(B/A_r)}$$

Example 35

There are two identical boxes containing respectively 4 white and 3 red balls, 3 white and 7 red balls. A box is chosen at random and a ball is drawn from it. Find the probability that the ball is white. If the ball is white, what is the probability that it is from first box?

Solution:

Let A_1, A_2 be the boxes containing 4 white and 3 red balls, 3 white and 7 red balls.

i.e

A_1	A_2
4 White 3 Red Total 7 Balls	3 White 7 Red Total 10 balls

One box is chosen at random out of two boxes.

$\therefore P(A_1) = P(A_2) = \frac{1}{2}$

One ball is drawn from the chosen box. Let B be the event that the drawn ball is white.

$\therefore P(B/A_1)$ = P(that the drawn ball is white from the Ist Box)

$\mathbf{P(B/A_1) = \frac{4}{7}}$

$\therefore P(B/A_2)$ = P (that the white ball drawn from the IInd Box)

$\Rightarrow P(B/A_2) = \frac{3}{10}$

P (B) = P (that the drawn ball is white)

$= P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)$

$= \frac{1}{2}\,\frac{4}{7} + \frac{1}{2}\,\frac{3}{10}$

$= \frac{61}{140}$

Now by Baye's Theorem, probability that the white ball comes from the Ist Box is,

$$P(B_1/A) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$= \frac{\frac{1}{2}\,\frac{4}{7}}{\frac{1}{2}\,\frac{4}{7} + \frac{1}{2}\,\frac{3}{10}} = \frac{\frac{4}{7}}{\frac{4}{7} + \frac{3}{10}} = \frac{40}{61}$$

Example 36

A factory has 3 machines A_1, A_2, A_3 producing 1000, 2000, 3000 bolts per day respectively. A_1 produces 1% defectives, A_2 produces 1.5% and A_3 produces 2% defectives. A bolt is chosen at random at the end of a day and found defective. What is the probability that it comes from machine A_1?

Solution:

$P(A_1)$ = P(that the machine A_1 produces bolts)

$= \frac{1000}{6000} = \frac{1}{6}$

$P(A_2)$ = P(that the machine A_2 produces bolts)

$= \frac{2000}{6000} = \frac{1}{3}$

$P(A_3)$ = P(that the machine A_3 produces bolts)

$= \frac{3000}{6000} = \frac{1}{2}$

Let B be the event that the chosen bolt is defective

$\therefore$ $P(B/A_1)$= P(that defective bolt from the machine A_1)

= .01

Similarly $P(B/A_2)$ = P(that the defective bolt from the machine A_2)

= .015 and

$P(B/A_3)$ = P(that the defective bolt from the machine A_3)

= .02

We haev to find $P(A_1/B)$

Hence by Baye’s theorem, we get

$$P(A_1/B) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)P(B/A_3)}$$

$$= \frac{\frac{1}{6} \text{x } (.01)}{\frac{1}{6} \text{ x } (.01) + \frac{1}{3} \text{ x } (.015) + \frac{1}{2} \text{x}(.02)}$$

$$= \frac{.01}{.01 + .03 + .06} = \frac{.01}{.1} = \frac{1}{10}$$

Example 37

In a bolt factory machines A_1, A_2, A_3 manufacture respectively 25%, 35% and 40% of the total output. Of these 5, 4, 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine A_2?

Solution:

$P(A_1)$ = P(that the machine A_1 manufacture the bolts)

$$= \frac{25}{100} = .25$$

Similarly $P(A_2) = \frac{35}{100} = .35$ and

$P(A_3) = = \frac{40}{100} = .4$

Let B be the event that the drawn bolt is defective.

$\therefore$ $P(B/A_1)$= P(that the defective bolt from the machine A_1)

$$= \frac{5}{100} = .05$$

Similarly $P(B/A_2) = \frac{4}{100} = .04$ and $P(B/A_3) = \frac{2}{100} = .02$

we have to find $P(A_2/B)$

Hence by Baye's theorem, we get

$$P(A_2/B) = \frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1) + P(A)P(B/A_2) + P(A_3)P(B/A_3)}$$

$$= \frac{(.35)\,(.04)}{(.25)\,(.05) + (.35)\,(.04) + (.4)\,(.02)}$$

$$= \frac{28}{69}$$

EXERCISES 10.3

1) Three coins are tossed. Find the probability of getting (i) no heads (ii) at least one head.

2) A perfect die is tossed twice. Find the probability of getting a total of 9.

3) A bag contains 4 white and 6 black balls. Two balls are drawn at random. What is the probability that (i) both are white (ii) both are black.

4) A number is chosen out of the numbers {1,2,3,....100} What is the probability that it is

   (i) a perfect square (ii) a multiple of 3 or 7.

5) A bag contains 4 white, 5 black, and 6 red balls. A ball is drawn at random. What is the probability that is red or white.

6) If two dice are thrown simultaneously, what is the probability that the sum of the points on two dice is greater than 10?

7) A person is known to hit the target 3 out of 4 shots where as another person is known to hit 2 out of 3 shots. Find the probability of the target being hit when they both shoot.

8) There are 3 boxes containing respectively 1 white, 2 red, 3 black balls; 2white, 3 red, 1 black ball : 2 white, 1 red, 2 black balls. A box is chosen at random and from it two balls are drawn at random. The two balls are 1 red and 1 white. What is the probability that they come from the second box?

9) In a company there are three machines A_1, A_2 and A_3. They produce 20%, 35% and 45% of the total output respectively. Previous experience shows that 2% of the products produced by machines A_1 are defective. Similarly defective percentage for machine A_2 and A_3 are 3% and 5% respectively. A product is chosen at random and is found to be defective. Find the probability that it would have been produced by machine A_3?

10) Let U_1, U_2, U_3 be 3 urns with 2 red and 1 black, 3 red and 2 black, 1 red and 1 black ball respectively. One of the urns is chosen at random and a ball is drawn from it. The colour of the ball is found to be black. What is the probability that it has been chosen from U_3?

EXERCISE 10.4

Choose the correct answer

1) Which one is the measure of central tendency

(a) Range (b) Coefficient of Variation

(c) Median (d) None of these

2) Arithmetic Mean of 2, -2 is

(a) 2 (b) 0 (c) -2 (d) None of these

3) Median for 2, 20, 10, 8, 1 is

(a) 20 (b) 10 (c) 8 (d) None of these

4) Mode is

(a) Most frequent value (b) Middlemost value

(c) First value of the series (d) None of these

5) The Geometric mean of 0,2, 8, 10 is

(a) 2 (b) 10 (c) 0 (d) None of these

6) For 'n' individual observation, the Harmonic mean is

a) $\sqrt{\frac{n}{\Sigma x}}$ (b) $\sqrt{\frac{\frac{n}{1}}{\Sigma \frac{1}{x}}}$ (c) $\frac{n}{\Sigma \frac{1}{x}}$ (d) None of these

7) Which of the following is not a measure of dispersion.

(a) H.M (b) S.D. (c) C.V. (d) None of these

8) If the mean and variance of a series are 10 and 25, then co-efficient of variation is

(a) 25 (b) 50 (c) 100 (d) None of these

9) If the S.D. and the C.V. of a series are 5 and 25, then the arithmetic mean is

(a) 20 (b) 5 (c) 10 (d) None of these

10) Probability that atleast one of the events A, B occur is

(a) $P(A \cup B)$ (b) $P(A \cap B)$ (c) $P(A/B)$ (d) None of these

11) $P(A) + P(\overline{A})$ is

(a) -1 (b) 0 (c) 1 (d) None of these

12) If A and B are mutually exclusive events, then $P(A \cup B)$ is

(a) $P(A) + P(B)$ (b) $P(A) + P(B) - P(A \cap B)$

(c) 0 (d) None of these

13) The probability of drawing any one spade card from a pack of cards is.

(a) $\frac{1}{52}$ (b) $\frac{1}{13}$ (c) $\frac{1}{4}$ d) None of these

14) The probability of drawing one white ball from a bag containing 6 red, 8 black and 10 yellow balls is

(a) $\frac{1}{52}$ (b) 0 (c) $\frac{1}{24}$ (d) None of these

[redacted]

(a) $\frac{P(A \cap B)}{P(A)}$ (b) $\frac{P(A \cap B)}{P(B)}$, $P(B) = 0$

(c) $\frac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$ (d) None of these

16) Which is based on all the observations?
(a) Range (b) Median (c) Mean (d) Mode

17) Which is not unduly affected by extreme item?
(a) Median (b) Mean (c) Mode (d) None of these

18) The emprical relation between mean, median and mode is
(a) Mean - mode = 3 median (b) Mean -mode=3 (mean -median)
(c) Mean - mode = 2 mean (d) mean = 3 median - mode

19) Square of S.D. is called
(a) mean deviation (b) quartile deviation
(c) variance (d) range

20) If A and B are independent event, then $P(A \cap B)$ is
(a) P(A) P(B) (b) P(A) + P(B) (c) P(A/B) (d) P(B) - P(A)

21) Which of the following is correct?
(a) H.M. $\leq$ G.M. $\leq$ A.M. (b) H.M. $\geq$ G.M. $\leq$ A.M.
(c) A.M. < G.M. < H.M. (d) None of these

22) Which of the following is correct?
(a) $(A.M. \times H.M.)^2$ (b) $A.M. \times H.M. = (G.M.)^2$
(c) $(H.M. \times G.M.) = (A.M.)^2$ (d) $\frac{A.M. + G.M.}{2} = H.M.$

23) Probability of sure event is
(a) 1 (b) 0 (c) -1 (d) S

24) Probability of an impossible event is
(a) 1 (b) 0 (c) 2 (d) ϕ

25) A single letter is selected at random from the word PROBABILITY The probability that it is a vowel is

(a) $\frac{3}{11}$ (b) $\frac{2}{11}$ (c) $\frac{4}{11}$ (d) 0

ANSWERS

MATRICES AND DETERMINANTS

Exercise 1.1

2) i) $A + B = \begin{pmatrix} 12 & 3 & 7 \\ 4 & 12 & 7 \\ 6 & -1 & 8 \end{pmatrix}$ ii) $\begin{pmatrix} 12 & 3 & 7 \\ 4 & 12 & 7 \\ 6 & -1 & 8 \end{pmatrix}$

iii) $5A = \begin{pmatrix} 15 & 5 & 10 \\ 20 & 45 & 40 \\ 10 & 25 & 23 \end{pmatrix}$ iv) $\begin{pmatrix} 18 & 4 & 10 \\ 0 & 6 & -2 \\ 8 & -12 & -4 \end{pmatrix}$

3) $AB = \begin{pmatrix} 8 & 4 \\ -9 & 12 \end{pmatrix}$, $BA = \begin{pmatrix} 14 & 16 \\ -3 & 6 \end{pmatrix}$

4) $AB = \begin{pmatrix} 11 & -40 & 39 \\ 0 & 18 & -14 \\ 7 & -18 & -15 \end{pmatrix}$, $BA = \begin{pmatrix} -8 & 38 & 3 \\ -4 & 14 & 1 \\ -9 & 41 & 8 \end{pmatrix}$

5) $AB = \begin{pmatrix} 9 & 13 \\ 12 & 18 \end{pmatrix}$, $BA = \begin{pmatrix} 7 & 16 & -10 \\ 17 & 16 & -6 \\ 8 & -1 & 4 \end{pmatrix}$

11) $AB = 29$, $BA = \begin{pmatrix} 12 & 20 & 24 \\ 3 & 5 & 6 \\ 6 & 10 & 12 \end{pmatrix}$

12) $AB = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $BA = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

13) Total requirement of calories and proteins for family A is 12000 and 320 respectively and for family B is 10900 and 295.

14) $\begin{pmatrix} 11 & 15 & 16 \\ 15 & 15 & 16 \\ 25 & 35 & 43 \end{pmatrix}$ 15) $\begin{pmatrix} -3 & -6 \\ -7 & 2 \end{pmatrix}$ 18) $\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$

22) (i) $\begin{pmatrix} 60 & 44 \\ 27 & 32 \end{pmatrix}$ (ii) $\begin{pmatrix} 58 & 40 \\ 31 & 34 \end{pmatrix}$ (iii) $\begin{pmatrix} 44 & 6 \\ -5 & 10 \end{pmatrix}$ (iv) $\begin{pmatrix} 32 & 19 \\ 0 & 18 \end{pmatrix}$

23) (i) $\begin{pmatrix} 45 & 60 & 55 & 30 \\ 58 & 72 & 40 & 80 \end{pmatrix}$ (ii) 2 x 4 (iii) $\begin{pmatrix} 45 & 58 \\ 60 & 72 \\ 55 & 40 \\ 30 & 80 \end{pmatrix}$

(iv) (i) is the transpose of (iii)

Exercise 1.2

1) (i) 24 (ii) 9 (iii) 8 2) 10 3) 1

4) |A| = 0, A is singular 5) A is non-singular

6) 0 7) 0 8) -120 9) 5

Exercise 1.3

1) (c)	2) (c)	3) (a)	4) (c)	5) (b)
6) (b)	7) (a)	8) (c)	9) (d)	10) (a)
11) (b)	12) (c)	13) (c)	14) (b)	15) (a)
16) (c)	17) (a)	18) (b)	19) (b)	20) (b)
21) (a)	22) (b)	23) (a)	24) (a)	25) (c)
26) (b)	27) (d)	28) (d)	29) (b)	30) (a)

ALGEBRA

Exercise 2.1

1) $\frac{4}{5(x-3)} + \frac{1}{5(x+2)}$ 2) $\frac{-19}{x+2} + \frac{21}{x+3}$ 3) $\frac{21}{x+3} + \frac{21}{x+3}$

4) $\frac{1}{2(x+2)} + \frac{1}{2(x-2)} - \frac{1}{x+1}$ 5) $\frac{-2}{25(x+3)} + \frac{2}{25(x-2)} + \frac{3}{5(x-2)^2}$

6) $\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$ 7) $\frac{1}{4(x-1)} - \frac{1}{4(x+1)} + \frac{1}{2(x+1)^2}$

8) $\frac{2}{x-1} - \frac{5}{(x+3)^2}$ 9) $\frac{4}{3x-2} + \frac{x-5}{x^2-2x-1}$ 10) $\frac{3}{2(x-1)} - \frac{3x+1}{2(x^2+1)}$

Exercise 2.2

1) n = 10 2) 21 3) (i) $\frac{13!}{3!\,3!\,3!}$ (ii) $\frac{11!}{2!\,2!\,2!}$ (iii) $\frac{11!}{4!\,4!\,2!}$ 4) 1344

5) 6666600 6) (i) 8! 4! (ii) (7!) ($^8p_4$) 7) 1440 8) 1440 9) (i) 720 (ii) 24

Exercise 2.3

1) (i) 210 (ii) 105 2) 16 3) 8 4) 780

5) 3360 6) 858 7) 9 8) 20790

Exercise 2.5

1) $\frac{n(n+1)(n+2)(n+3)}{4}$ 2) $\frac{n(n+1)(n+2)(3n+5)}{12}$

3) $\frac{2n(n+1)(2n+1)}{3}$ 4) $n(3n^2+6n+1)$

5) $\frac{n}{3}(2n^2+15n+74)$ 6) $\frac{n(n+1)(n+2)}{6}$

Exercise 2.6

1) ${}^{11}c_5(-2)^5x$, ${}^{11}c_6\frac{2^6}{x}$ 2) ${}^{12}c_6\frac{y^3}{x^3}$

3) ${}^{10}c_4(256)$ 4) $\frac{144x^2}{y^7}$

5) ${}^{9}c_4\frac{3x^{17}}{16}$, $-{}^{9}c_5\frac{x^{19}}{96}$ 6) ${}^{12}c_4(2^4)$

Exercise 2.7

1) (a)	2) (a)	3) (b)	4) (b)	
5) (a)	6) (a)	7) (a)	8) (b)	
9) (c)	10) (a)	11) (a)	12) (a)	
13) (a)	14) (b)	15) (c)	16) (b)	17) (d)

SEQUENCES AND SERIES

Exercise 3.1

1) $\frac{4}{23}$, $\frac{2}{19}$ 2) $\frac{1}{248}$

Exercise 3.2

1) 11, 17, 23 2) 15, 45, 135, 405, 1215

3) $\frac{1}{8}$, $\frac{1}{11}$, $\frac{1}{14}$, $\frac{1}{17}$ 4) 4, 64

Exercise 3.4

1) (a) 2, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{5}{24}$, $\frac{1}{20}$ (b) $\frac{1}{2}$, $-\frac{1}{3}$, $\frac{1}{4}$, $-\frac{1}{5}$, $\frac{1}{6}$

c) 1, $\frac{1}{4}$, $\frac{1}{27}$, $\frac{1}{256}$, $\frac{1}{3125}$ (d) 1, 0, $\frac{1}{2}$, 0, $\frac{1}{3}$

(e) 2, 16, 96, 512, 2560 (f) -1, 1, -1, 1, -1

(g) 5, 11, 17, 23, 29

2) 2, 6, 3, 9, 4, 12, 5 3) (a) {0, 2} b) {-1, 1}

4) (a) n^2 (b) 4n-1 (c) $2+\frac{1}{10^n}$ (d) n^2-1 (e) $\frac{10\ n}{3^n}$

5) (a) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$ (b) 5, -10, 20, -40, 80, -160

(c) 1, 4, 13, 40, 121, 364 (d) 2, 6, 15, 34, 73, 152

(e) 1, 5, 14, 30, 55, 91 (f) 2, 1, 0, -1, -2, -3

(g) 1, 1, 3, 11, 123, 15131 (h) 1, -1, 3, 1, 5, 3

Exercise 3.5

1) Rs. 27,350 2) i) Rs. 5,398 ii) Rs. 5,405 3) Rs. 95, 720

4) Rs. 13,110 5) Rs. 1,710 6) Rs. 8,000 7) 12%

8) 22 $\frac{1}{2}$ years (nearly) 9) 16.1% 10) 12.4%

Exercise 3.6

1) Rs. 5,757.14 2) Rs. 2,228 3) Rs. 6,279 4) Rs. 3,073

5) Rs. 12,590 6) Machine B may be purchased 7) Rs. 1,198

8) Rs. 8,097 9) Rs. 5,796 10) Rs. 6,987 11) Rs. 46,050

12) Rs. 403.40 13) Rs. 7,398

Exercise 3.7

1) (a)	2) (a)	3) (b)	4) (d)	5) (a)	6) (b)
7) (b)	8) (a)	9) (a)	10) (b)	11) (d)	12) (a)
13) (a)	14) (c)	15) (d)	16) (a)	17) (b)	18) (b)
19) (b)	20) (a)	21) (a)	22) (d)	23) (b)	24) (b)
25) (b)	26) (b)	27) (b)	28) (c)	29) (d)	30) (a)
31) (b)	32) (c)	33) (b)			

ANALYTICAL GEOMETRY

Exercise 4.1

1) $8x+6y-9 = 0$
2) $x-4y-7 = 0$
3) $8x^2+8y^2-2x-36y+35 = 0$
4) $x^2+y^2-6x-14y+54 = 0$
5) $3x-4y = 12$
6) $x^2-3y^2-2y+1 = 0$
7) $x-y-6 = 0$
8) $24x^2-y^2 = 0$
9) $3x^2+3y^2+2x+12y-1 = 0$
10) $2x+y-7 = 0$

Exercise 4.2

1) $2x-3y+12 = 0$ 2) $x-y+5\sqrt{2} = 0$
3) $x + 2y - 6 = 0$; $2x + y = 0$
4) $\frac{7}{5}$ 5) $-\frac{3}{2}$ or $\frac{17}{6}$ 6) $2x-3y+12 = 0$ 7) $x-\sqrt{3}\,y + 2 + 3\sqrt{3} = 0$
8) $9x - 33y + 16 = 0$; $77x + 21y - 22 = 0$

Exercise 4.3

2) $k = -33$
3) $4x-3y+1 = 0$
4) $x-2y+2 = 0$
5) $3x+y-5 = 0$
6) Rs. 0.75
7) $y = 7x+500$
8) $y = 4x+6000$
9) $2y = 7x+24000$

Exercise 4.4

1) $x^2+y^2+8x+4y-16 = 0$
2) $x^2+y^2-4x-6y-12 = 0$
3) π, $\frac{p}{4}$
4) $x^2+y^2+8x-12y-33 = 0$
5) $x^2+y^2-8x+2y-23 = 0$
6) $x^2+y^2-6x-6y+13 = 0$
7) $x^2+y^2-6x-8y+15 = 0$
8) $5x^2+5y^2-26x-48y+24 = 0$
9) $x^2+y^2-4x-6y-12 = 0$

Exercise 4.5

1) $x+3y-10 = 0$
2) $2x+y-7 = 0$
3) 6 units
4) $a^2(l^2+m^2) = n^2$
6) $\frac{1}{2}\sqrt{46}$

Exercise 4.6

1) (a)	2) (b)	3) (a)	4) (b)	5) (b)	6) (b)
7) (c)	8) (c)	9) (b)	10) (b)	11) (a)	12) (c)
13) (b)	14) (a)	15) (b)	16) (b)	17) (a)	

TRIGONOMETRY

Exercise 5.1

12) $\frac{31}{12}$ 13) $\frac{1}{8}$ 14) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ 18) $\frac{3}{4}$ 19) $1 \pm \sqrt{2}$

Exercise 5.2

3) $\cos A = \frac{24}{25}$, $\operatorname{cosec} A = \frac{-25}{7}$ 4) $\frac{-1331}{276}$ 5) 1 6) cotA

8) (i) -cosec 23^o (ii) cot 26^o

Exercise 5.3

5. (i) $-(2+\sqrt{3})$ (ii) $\frac{2\sqrt{2}}{1-\sqrt{3}}$ 8) (i) $\frac{36}{325}$ (ii) $-\frac{253}{325}$

Exercise 5.4

14) $\sin 3A = \frac{117}{125}$ $\cos 3A = \frac{-44}{125}$; $\tan 3A = \frac{-117}{44}$

Exercise 5.5

1. (i) $\frac{1}{2}$ ($\cos\frac{A}{2}$ - cosA) (ii) $\frac{1}{2}$ (cos2C - cos2B)

   (iii) $\frac{1}{2}$ ($\frac{1}{2}$ + cos2A) (iv) $\frac{1}{2}$ (cos3A + $\cos\frac{A}{3}$)

2. (i) $2\cos 42^o \sin 10^o$ (ii) - 2sin4Asin2A (iii) $\cos 20^o$

Exercise 5.6

1) (i) $\frac{\pi}{6}$ (ii) $5\frac{\pi}{6}$ (iii) $3\frac{\pi}{4}$ (iv) $\frac{\pi}{6}$ (v) $-\frac{\pi}{4}$ (vi) $\frac{\pi}{4}$

2) (i) $\theta = n\pi \pm \frac{p}{3}$; $n \in Z$ (ii) $\theta = 2n\pi \pm \frac{p}{3}$; $n \in Z$, $\theta = 2n\pi \pm \frac{2\pi}{3}$; $n \in Z$

   (iii) $\theta = n\pi \pm \frac{p}{2}$; $n \in Z$ iv) $\theta = n\pi \pm \frac{p}{3}$; $n \in Z$

Exercise 5.7

6) x = -1 or $\frac{1}{6}$ 7) x = $\frac{1}{2}$ or -4 9) $\frac{33}{65}$

Exercise 5.8

1) (d)	2) (a)	3) (c)	4) (a)	5) (c)	6) (a)
7) (b)	8) (d)	9) (b)	10) (c)	11) (c)	12) (b)
13) (c)	14) (a)	15) (d)	16) (c)	17) (c)	18) (b)
19) (d)	20) (a)	21) (c)	22) (c)	23) (c)	24) (c)
25) (a)	26) (a)	27) (b)	28) (c)	29) (a)	30) (d)
31) (c)	32) (a)	33) (b)	34) (a)	35) (d)	36) (d)
37) (a)	38) (a)	39) (a)	40) (b)		

FUNCTIONS AND THEIR GRAPHS

Exercise 6.1

5) $2x-3+h$ 6) 0 7) Domain $\{ x / < 0 \text{ or } x \geq 1\}$

8) $C = \begin{cases} 100n & ; 0 \leq n < 25 \\ 115n - \frac{n^2}{25} & ; 25 \leq n \end{cases}$ 9) $(-\infty, 2]$ and $[3, \infty]$

12) $f\left(\frac{1}{x}\right) = \frac{1-x}{3+5x}$, $\frac{1}{f(x)} = \frac{3x+5}{x-1}$ 13) $2\sqrt{x^2+1}$; ± 2

Exercise 6.2

4) log 8 ; $(\log 2)^3$

5) (i) 1 (ii) -11 (iii) -5 (iv) -1 (v) $41-29\sqrt{2}$

(vi) 0.25 (vii) 0 (viii) $\frac{8}{3}$; domain is $R-\{-\frac{1}{2}\}$

6) (i) 1, 1 (ii) -1, 1 (iii) $\frac{1}{2}$, $-\frac{1}{2}$

(iv) (0, 0) ; The domain is $R-\{(4n\pm1)\frac{\pi}{2}$; n is an integer$\}$

7) (i) $R - \{(2n\pm1)\pi ; n\in Z\}$ (ii) $R-\{2n\pi ; n\in Z\}$

(iii) $R-\{n\pi\pm\frac{\pi}{4} ; n\in Z\}$ (iv) R

(v) $R-\{2n\pi ; n\in Z\}$ (vi) $R-\{(2n+1)\frac{\pi}{2} ; n\in Z\}$

8) Rs. 1,425 9) 74 years

10) i) $f(x) = \frac{1}{3}x + \frac{10}{3}$ ii) $f(3) = \frac{13}{3}$ (iii) a = 290

Exercise 6.3

1) (d)	2) (d)	3) (a)	4) (a)	5) (a)	6) (c)
7) (b)	8) (c)	9) (b)	10) (c)	11) (d)	12) (a)
13) (a)	14) (b)	15) (b)			

DIFFERENTIAL CALCULUS

Exercise 7.1

1) (i) 10/3 (ii) - 5 (iii) 1/3 (iv) - 1/√2 (v) 2

(vi) 1 (vii) $\frac{15}{8}a^{7/24}$ (viii) 5/3 (ix) 1 (x) 4

(xi) 12 (xii) 5 / 2

2) 5 4) 28 / 5 , f (2) does not exist.

Exercise 7.2

2) 5/4, - 4/3. (6) x = 3 and x = 4

Exercise 7.3

1) (i) - sin x (ii) $\sec^2 x$ (iii) - cotx cosec x (iv) $\frac{1}{2\sqrt{x}}$

2) (i) $12x^3 - 6x^2 + 1$ (ii) $\frac{-20}{x^5} + \frac{6}{x^4} - \frac{1}{x^2}$

(iii) $\frac{1}{2\sqrt{x}} - \frac{1}{3x^{2/3}} + e^x$ (iv) $\frac{-1}{x^2}\left(3 + x^2\right)$

(v) $\sec^2 x + 1/x$ (vi) $x^2 e^x (x + 3)$

(vii) $\frac{15}{2}x^{3/2} - 6x^{1/2} - x^{-3/2}$ (viii) $\frac{n}{x^{n+1}}\left(ax^{2n} - b\right)$

(ix) $2x(6x^2 + 1)$ (x) $x^2 \cos x + 2(\cos x + x \sin x)$

(xi) $\sec x(1 + 2\tan^2 x)$ (xii) $2\sin x(x - 1) + x\cos(x - 2) + e^x$

(xiii) $2x(2x^2 + 1)$ (xiv) $x^{n-1}(1 + n\log x)$

(xv) $2(x\tan x + \cot x) + x(x\sec^2 x - 2\,\mathrm{cosec}^2 x)$

(xvi) $\frac{\sec x}{2\sqrt{x}}(2x\tan x + 1)$ (xvii) $\frac{e^x}{\left(1 + e^x\right)^2}$

(xviii) $\tan\frac{x}{2}\left(1+\tan^2\frac{x}{2}\right)$ (xix) $\frac{-30}{(3+5x)^2}$

(xx) $\frac{x^2-1}{x^2-4}$ (xxi) $1-\frac{1}{x^2}$

(xxii) x (1 + 2 log x) (xxiii) x $\sec^2$ x + tan x – sin x

(xxiv) $\frac{xe^x}{(1+x)^2}$

Exercise 7.4

1) $\frac{3x-1}{\sqrt{3x^2-2x+2}}$ 2) $\frac{-10}{3(8-5x)^{1/3}}$ 3) $e^x \cos(e^x)$

4) $e^{\sec x}$ (secx tanx) 5) tan x 6) $2xe^{x^2}$

7) $\frac{1}{\sqrt{x^2+1}}$ 8) - 3 sin(3x-2) 9) -2x $\tan(x^2)$

10) $\frac{2(x^2-3)}{x^2-4}$ 11) $e^{\sin x+\cos x}$ (cosx – sinx)

12) - cosec^2 x. $e^{\cot x}$ 13) $\frac{1}{1+e^x}$ 14) 2 cotx

15) $\frac{1}{2\sqrt{\tan x}}(e^{\sqrt{\tan x}}\sec^2 x)$ 16) 2x $\cos x^2$ 17) $\frac{n[\log(\log(\log x))]^{n-1}}{x.\log x.\log(\log x)}$

18) - 2 sin 2x 19) $\frac{1}{1+e^x}-\frac{\log(1+e^x)}{e^x}$ 20) $\frac{4x}{1-x^4}$

21) $\frac{1}{3}(x^3+x+1)^{-2/3}(3x^2+1)$ 22) $\frac{\cos(\log x)}{x}$

23) $x^{\log(\log x)}$ [1+log (logx) 24) 18 x ($3x^2$ + 4$)^2$

Exercise 7.5

1) $\frac{3}{\sqrt{1-x^2}}$ 2) $\frac{3}{1+x^2}$ 3) $\frac{2}{1+x^2}$ 4) $\frac{2}{1+x^2}$ 5) $\frac{2}{1+x^2}$

6) $\frac{1}{2(1+x^2)}$ 7) $\frac{1}{2(1+x^2)}$ 8) $\frac{1}{\sqrt{a^2-x^2}}$ 9) x^x (1 + logx)

$\frac{b}{a}\cos ec\, q$

10) $(\sin x)^{\log x}\left[\cot x.\log x+\frac{\log\sin x}{x}\right]$ 11) $x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}\right]$

12) $(3x-4)^{x-2}\left[\frac{3(x-2)}{3x-4}+\frac{\log(3x-4)}{x-2}\right]$ 13) $e^{x^x}.x^x(1+\log x)$

14) $x^{\log x}\left(\frac{2\log x}{x}\right)$ 15) $\frac{5}{3}.\sqrt[3]{\frac{4+5x}{4-5x}}\left[\frac{8}{16-25x^2}\right]$

16) $(x^2+2)^5(3x^4-5)^4\left[\frac{10x}{x^2+2}+\frac{48x^3}{3x^4-5}\right]$ 17) $x^{1/x}\left[\frac{1}{x^2}(1-\log x)\right]$

18) $(\tan x)^{\cos x}$ (cosec x – sin x log tanx)

19) $\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]$ 20) $\frac{2x}{\sqrt{1+x^2}\,(1-x^2)^{3/2}}$

21) $\frac{x^3\sqrt{x^2+5}}{(2x+3)^2}\left[\frac{3}{x}+\frac{x}{x^2+5}-\frac{4}{2x+3}\right]$ (22) $a^x \quad \log a$

23) $x^{\sqrt{x}}\left(\frac{2+\log x}{2\sqrt{x}}\right)$ (24) $(\sin x)^x$ [x cot x + log sin x]

Exercise 7.6

1) $\frac{2a}{y}$ 2) $\frac{-x}{y}$ 3) $\frac{-y}{x}$ 4) $\frac{-b^2x}{a^2y}$ 5) $\frac{b^2x}{a^2y}$

6) $\frac{-(ax+hy)}{(hx+by)}$ 7) 1 8) $\frac{-x(2x^2+y^2)}{y(x^2+2y^2)}$ 9) $\frac{-\sqrt{y}}{\sqrt{x}}$

10) $\frac{y}{x}\left[\frac{x\log y-y}{y\log x-x}\right]$ 11) $-\frac{2x+1}{2y+1}$ 12) $-\frac{\sin(x+y)}{1+\sin(x+y)}$

13) $\frac{\log x}{(1+\log x)^2}$ 14) $\frac{\log\sin y+y\tan x}{\log\cos x-x\cot y}$ 15) $\frac{y-2x}{2y-x}$

Exercise 7.7

1) $-\frac{b}{a}\cot\theta$ 2) $-\frac{1}{t^2}$ 3) $\frac{a}{b}\operatorname{cosec}\theta$ 4) $\frac{1}{t}$

5) $-\tan\theta$ 6) $t\cos t$ 7) $\tan\theta$ 8) $\frac{2(t^2-1)}{t^{3/2}}$

9) $\frac{t \quad \tan t}{\sin(\log t)}$ 10) -1 11) $\frac{1}{t}$

Exercise 7.8

1) 32 2) a^2y 3) $-\frac{1}{(1+x)^2}$ 4) $-\frac{1}{2at^3}$

5) $-\frac{b}{a^2}\operatorname{cosec}^3\theta$ 6) $\frac{1}{3a}\sec^4\theta \operatorname{cosec}\theta$ 11) $-\frac{1}{x^2}$

Exercise 7.9

1) (c)	2) (b)	3) (d)	4) (a)	5) (d)	6) (c)
7) (c)	8) (b)	9) (c)	10) (a)	11) (c)	12) (c)
13) (a)	14) (d)	15) (a)	16 (b)	17) (b)	18) (d)
19) (a)	20) (b)	21) (b)	22) (c)	23) (c)	24) (b)
25) (a)	26) (b)	27) (c)	28) (c)	29) (a)	30) (b)
31) (b)	32) (b)	33) (a)	34) (c)	35) (a)	36) (b)
37) (c)	38) (d)	39) (d)	40) (b)	41) (c)	42) (a)
43) (a)	44) (b)				

INTEGRAL CALCULUS

Exercise 8.1

(1) $x(x^3 - 1) + C$ (2) $x^5 + \frac{2}{3}x\sqrt{x} - 14\sqrt{x} + C$

(3) $\frac{x^4}{2} + 4x^2 + 5\log x + e^x + C$ (4) $\frac{x^2}{2} + \log x + 2x + C$

(5) $\frac{x^4}{4} - \frac{1}{2x^2} + \frac{3}{2}x^2 + 3\log x + C$ (6) $5\sec x - 2\cot x + C$

(7) $\frac{2}{7}x^{7/2} + \frac{2}{5}x^{5/2} + \log x + C$ (8) $\frac{2}{7}x^{7/2} + \frac{6}{5}x^{5/2} + 8x^{1/2} + C$

(9) $3e^x + 2\sec^{-1}(x) + C$

(10) $\log x - \dfrac{1}{3x^3} + C$

(11) $9x - \dfrac{4x^3}{3} + C$

(12) $\dfrac{2}{3}x^{3/2} + \dfrac{2}{5}x^{5/2} + x^2 + C$

(13) $\dfrac{3}{2}x^{2/3} + 3\sin x + 7\cos x + C$

(14) $2x^{1/2} - \dfrac{2}{3}x^{3/2} + +C$

(15) $\dfrac{2}{3}x\sqrt{x+3} + C$

(16) $\dfrac{2}{3}(x+7)\sqrt{x+1} + C$

(17) $x - 2\tan^{-1} x + C$

(18) $x - \tan^{-1} x + C$

(19) $(\sin x + \cos x) + C$

(20) $\tan\dfrac{x}{2} + C$

(21) $-\dfrac{1}{3x^3} + e^{-x} + C$

(22) $\log x + e^{-x} + C$

(23) $\log x + \dfrac{1}{x} + e^x + C$

(24) $3x^3 + 4x^2 + 4x + C$

(25) $-\dfrac{1}{x} - 2e^{-2x} + 7x + C$

(26) $\tan x + \sec x + C$

Exercise 8.2

(1) $\dfrac{1}{12(2-3x)^4} + C$

(2) $\dfrac{1}{2(3-2x)} + C$

(3) $\dfrac{5}{24}(4x+3)^{6/5} + C$

(4) $\dfrac{e^{4x+3}}{4} + C$

(5) $\dfrac{2}{3\sqrt{x-1}}(x^2 + 4x + 8) + C$

(6) $\dfrac{1}{2}(x^3 + x - 4)^2 + C$

(7) $-\dfrac{1}{2}\cos(x^2) + C$

(8) $-2\cos\sqrt{x} + C$

(9) $\dfrac{1}{3}(\log x)^3 + C$

(10) $\dfrac{2}{3}(x^2 + x)^{3/2} + C$

(11) $\sqrt{x^2 + 1} + C$

(12) $\dfrac{1}{8}(x^2 + 2x)^4 + C$

(13) $\log(x^3 + 3x + 5) + C$

(14) $\dfrac{1}{6}\tan^{-1}\left(\dfrac{x^3}{2}\right) + C$

(15) $\log(e^x + e^{-x}) + C$

(16) $\log(\log x) + C$

(17) $\tan(\log x) + C$

(18) $-\dfrac{1}{4(2x+1)^2} + C$

(19) $\log\{\log(\log x)\} + C$

(20) $\dfrac{1}{6(1-2\tan x)^3} + C$

(21) $\log(\sin x) + C$

(22) $-\log(\operatorname{cosec} x + \cot x) + C$

(23) $\log(1 + \log x) + C$

(24) $\dfrac{1}{4}\{\tan^{-1}(x^2)\}^2 + C$

(25) $\dfrac{2}{3}(3 + \log x)^{3/2} + C$

(26) $\dfrac{1}{4}\log\dfrac{x^4}{x^4+1} + C$

(27) $(\tan\sqrt{x})^2 + C$

(28) $\dfrac{(2x+4)^{3/2}}{3} + C$

(29) $\dfrac{(x^2-1)^5}{5} + C$

(30) $\dfrac{2}{3}(x^2 + x + 4)^{3/2} + C$

(31) $\dfrac{1}{b}\log(a + b\tan x) + C$

(32) $\log\sec x + C$

Exercise 8.3

1) $\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) + C$

2) $\dfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x) + C$

3) $\dfrac{1}{4}\log\left(\dfrac{x-2}{x+2}\right) + C$

4) $\dfrac{1}{2\sqrt{5}}\log\left(\dfrac{\sqrt{5}+x}{\sqrt{5}-x}\right) + C$

5) $\dfrac{1}{3}\log\left(3x + \sqrt{9x^2-1}\right) + C$

6) $\dfrac{1}{6}\log\left(6x + \sqrt{36x^2+25}\right) + C$

7) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{2x}{3}\right) + C$

8) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right) + C$

9) $\dfrac{1}{6}\tan^{-1}\left(\dfrac{3x+1}{2}\right) + C$

10) $\log\left\{(x+2) + \sqrt{x^2+4x+2}\right\} + C$

11) $\log\left\{\left(x-\dfrac{1}{2}\right) + \sqrt{3-x+x^2}\right\} + C$

12) $\dfrac{1}{2}\log(x^2+4x-5) - \dfrac{1}{6}\log\left(\dfrac{x-1}{x+5}\right) + C$

13) $\frac{7}{2}\log\left(x^2 - 3x + 2\right) + \frac{9}{2}\log\left(\frac{x-2}{x-1}\right) + C$

14) $\frac{1}{2}\log\left(x^2 - 4x + 3\right) + 2\log\left(\frac{x-3}{x-1}\right) + C$ 15) $2\sqrt{2x^2 + x - 3} + C$

16) $2\sqrt{x^2 + 2x - 1} + 2\log\left\{(x+1) + \sqrt{x^2 + 2x - 1}\right\} + C$

Exercise 8.4

1) $-e^{-x}(x+1) + C$ 2) $\frac{x^2}{2}\left(\log x - \frac{1}{2}\right) + C$ 3) $x(\log x - 1) + C$

4) $\frac{a^x}{\log_e a}\left(x - \frac{1}{\log_e a}\right) + C$ 5) $x(\log x)^2 - 2x(\log x - 1) + C$

6) $-\frac{1}{x}(\log x + 1) + C$ 7) $\frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + C$

8) $\frac{\sin 3x}{9} - \frac{x\cos 3x}{3} + C$ 9) $x\cos^{-1}x - \sqrt{1-x^2} + C$

10) $x\tan^{-1}x - \frac{1}{2}\log\left(1+x^2\right) + C$ 11) $x\sec x - \log(\sec x + \tan x) + C$

12) $e^x(x^2 - 2x + 2) + C$

Exercise 8.5

1) $\frac{x}{2}\sqrt{x^2 - 36} - 18\log\left(x + \sqrt{x^2 - 36}\right) + C$

2) $\frac{x}{2}\sqrt{16 - x^2} + 8\sin^{-1}\left(\frac{x}{4}\right) + C$

3) $\frac{x}{2}\sqrt{x^2 + 25} + \frac{25}{2}\log\left(x + \sqrt{25 + x^2}\right) + C$

4) $\frac{x}{2}\sqrt{x^2 - 25} - \frac{25}{2}\log\left(x + \sqrt{x^2 - 25}\right) + C$

5) $\frac{x}{2}\sqrt{4x^2 - 5} - \frac{5}{4}\log\left(2x + \sqrt{4x^2 - 5}\right) + C$

6) $\frac{x}{2}\sqrt{9x^2 - 16} - \frac{8}{3}\log\left(3x + \sqrt{9x^2 - 16}\right) + C$

Exercise: 8.6

1) $\frac{29}{6}$ 2) 5 log 2 3) $\frac{p}{4}$ 4) $\frac{1}{\log_e 2}$

5) 3 (e – 1) 6) $\frac{1}{2}(e-1)$ 7) $\tan^{-1}(e) - \frac{p}{4}$ 8) $1 - \frac{p}{4}$

9) $\frac{p}{8}$ 10) $\frac{p}{2} - 1$ 11) (log 4)-1 12) $\frac{8}{3}\left(3\sqrt{3} - 1\right)$

13) $\frac{p}{4}$ 14) $\log\left(\frac{4}{3}\right)$ 15) $\sqrt{2}$ 16) $\frac{2}{3}$

17) $\frac{p}{2}$ 18) $\frac{1}{4}(e - 1)$

Exercise 8.7

1) $\frac{3}{2}$ 2) e - 1 3) $\frac{15}{4}$ 4) $\frac{1}{3}$

Exercise 8.8

1) (b)	2) (d)	3) (c)	4) (a)	5) (b)	6) (c)
7) (a)	8) (b)	9) (a)	10) (b)	11) (a)	12) (b)
13) (a)	14) (a)	15) (c)	16) (a)	17) (d)	18) (b)
19) (a)	20) (d)	21) (a)	22) (c)	23) (a)	24) (d)
25) (c)	26) (a)	27) (d)	28) (a)	29) (b)	30) (c)
31) (b)	32) (d)	33) (a)	34) (d)	35) (a)	

STOCKS, SHARES AND DEBENTURES

Exercise 9.1

1) Rs. 750 2) Rs. 1,000 3) 100 4) Rs. 7,200 5) Rs. 1,500

6) Rs. 9,360 7) $6\frac{2}{3}$ % 8) 15% 9) 12.5% 10) 20%

11) $7\frac{9}{13}$ % 12) 5% stock at 95 13) 18% debenture at 110

14) $13\frac{1}{3}$ % 15) Rs. 40,500 16) Rs. 160 17) Rs. 130

18) Rs. 675 19) Rs. 525 20) 2% 21) Rs. 5,500 22) Rs. 900, Rs. 90

23) Decrease in income Rs. 333.33 24) Rs. 120

25) Rs. 10,000, Rs. 24,000 26) 5% 27) 17.47%

Exercise 9.2

1) (b)	2) (b)	3) (a)	4) (a)	5) (a)	6) (d)
7) (b)	8) (a)	9) (a)	10) (d)	11) (b)	12) (a)

STATISTICS

Exercise 10.1

1) 29.6	2) 13.1	3) 4	4) 58	5) 33
6) 49.3	7) 34	8) 59.5	9) 20	10) 8
11) 48.18	12) 44.67	13) 69	14) 32	15) 13
16) 26.67	17) 183.35	18) 17.07	19) 28.02	20) 4.38
21) 8.229	22) 30.93			

Exercise 10.2

1) (a) 11, .58 (b) 29, .39 2) 12, .0896

3) 40, .33 4) S.D = 2.52 5) S.D = 3.25

6) (i) S.D = 13.24 (ii) S.D = 13.24 (iii) 13.24

7) S.D = 1.07 8) S.D = 1.44 9) S.D = 2.47

10) S.D = Rs. 31.87 (Crores) 11) C.V = 13.92

12) C.V(A) = .71, C.V(B) = .67 Since C.V(B) < C.V(A), CityB's price was more stable.

13) C.V =(x) = 5.24, C.V(y) = 1.90, since C.V(y) < C.V(x) City y's share was more stable.

Exercise 10.3

1) $\frac{1}{8}$, $\frac{7}{8}$	2) $\frac{1}{9}$	3) $\frac{2}{15}$, $\frac{1}{3}$	4) $\frac{1}{10}$, $\frac{43}{100}$	5) $\frac{2}{3}$
6) $\frac{1}{12}$	7) $\frac{11}{12}$	8) $\frac{6}{11}$	9) $\frac{45}{74}$	10) $\frac{15}{37}$

Exercise 10.4

1) (c)	2) (b)	3) (c)	4) (a)	5) (c)	6) (c)
7) (a)	8) (b)	9) (a)	10) (a)	11) (c)	12) (a)
13) (c)	14) (b)	15) (c)	16) (c)	17) (a)	18) (b)
19) (c)	20) (a)	21) (a)	22) (b)	23) (a)	24) (b)
25) (c)					

LOGARITHMS

Mean Differences

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1594	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	5200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	7	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	9702	9712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

LOGARITHMS

Mean Differences

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	6	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	4	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8912	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9764	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	4	4

ANTI-LOGARITHMS

Mean Differences

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	2	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	3	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	3	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2648	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	2	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	6	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	7

ANTI-LOGARITHMS

Mean Differences

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	2	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	7	7	8
.61	4074	4083	4093	4102	4111	4121	4030	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	8	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	6	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	12	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6566	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	14
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	14	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	10	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	14	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	7	9	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	21